Appendix
Printone: Interactive Resonance Simulation for Free-form Print-wind Instruments Design

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A Boundary Formulation of Acoustics

To make the paper self-contained, we briefly explain the boundary formulation of the Helmholtz equation. We refer readers to the book [2] for more details of the BEM implementation. The Helmholtz equation (1) has a kernel

$$G(\mathbf{x},\mathbf{y}) = \frac{\exp(+ikr)}{4\pi r}, \quad \text{where } r = ||\mathbf{x}-\mathbf{y}||, \tag{A.1}$$

which is the fundamental solution to the Dirac delta function $\delta(\mathbf{x}-\mathbf{y})$. Using this kernel function and the Neumann boundary condition, the second Stoke's theorem leads to the equation which the sound pressure on the surface $p(\mathbf{x})$ needs to satisfy

$$\frac{\Omega(\mathbf{x})}{4\pi}p(\mathbf{x}) + \int_S \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} p(\mathbf{y}) ds(\mathbf{y}) = G(\mathbf{x},\mathbf{x}_{src}), \quad \mathbf{x} \in S, \tag{A.2}$$

where the $\partial G(\mathbf{x},\mathbf{y})/\partial\mathbf{n}(\mathbf{y})$ derivative the kernel with respect to change of $\mathbf{y} \in S$ in the normal direction of the surface is

$$\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} = \frac{\exp(+ikr)}{4\pi r^2}(1-ikr)\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}}{||\mathbf{x}-\mathbf{y}||}. \tag{A.3}$$

The $\Omega(\mathbf{x})$ is a solid angle which takes 2π on a smooth surface, and is computed for triangle mesh using a formula presented in [3]. In our implementation, the sound pressure is stored at the vertices of a triangle mesh and linearly interpolated over the triangle faces. We discretize equation (A.2) using a typical collocation method, which formulates a linear system (3) by satisfying the equation at every vertex. We use a fifth-order Gaussian quadrature to compute this surface integration.

Once the reflection pressure at the vertices $\mathbf{p}$ in (3) is solved, the pressure value at the observation point $\mathbf{x}_{obs}$ inside medium Ω is computed with the surface integration

$$p(\mathbf{x}_{obs}) = -\int_S \frac{\partial G(\mathbf{x}_{obs},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} p(\mathbf{y}) ds(\mathbf{y}) + G(\mathbf{x}_{obs},\mathbf{x}_{src}), \quad \mathbf{x}_{obs} \in \Omega. \tag{A.4}$$

Our implementation is specifically categorized as the conventional boundary integration method (CBIM), in contrast to a more sophisticated model such as the Burton-Miller method [1]. The CBIM often suffers from errors in the frequency where the complementary region of the media $\bar{\Omega} = \{\mathbf{x} \in \mathbb{R}^3 | \mathbf{x} \notin \Omega\}$ has a fictitious resonance mode. In our simulation the complementary region $\bar{\Omega}$ is the solid region of the musical instrument. Since our complementary region $\bar{\Omega}$ is small compared to the cavity, the fictitious resonance mode is much higher compared to the fundamental cavity resonance frequency, and thus CBIM is adequate.

The off-diagonal (i, j)-entry of the resulting coefficient matrix A_{ij} is approximately written as:

$$A_{ij} \simeq \left[\frac{\mathbf{r}_{ij} \cdot \mathbf{n}_i}{4\pi r_{ij}^3} \Delta_j \right] \underbrace{\exp(+ikr_{ij})\,(1 - ikr_{ij})}_{g(\gamma)}, \tag{A.5}$$

where $\mathbf{r}_{ij}$ is a vector between i- and j-vertices, $r_{ij} = ||\mathbf{r}_{ij}||$, the $\mathbf{n}_i$ is the unit normal vector, Ω_i is the solid angle at the i-vertex, and Δ_j is one third of the area of triangles around j-vertex. Notice the nonlinearity of the coefficient matrix with respect to wavenumber k (see Sec. 5.1). Furthermore, the nonlinear dependent part $g(\gamma)$ is a function of $\gamma = kr_{ij}$ and if it is small, the linear approximation over the wavenumber is reasonable (see Sec. 6.2). Finally, the entry is invariant under the scaling geometry with s and scaling the wave number with $1/s$ i.e., $r_{ij} \to sr_{ij}$ and $k \to k/s$ (see Sec. 8).

B The Minimum Eigenvalue Bounds the Magnitude of System's Output from Below

Here, we show that if the minimum eigenvalue of the system's coefficient matrix $\mathbf{A} \in \mathbb{C}^{N \times N}$ is very small, the magnitude of the output $\mathbf{p} = \mathbf{A}^{-1}\mathbf{f}$ becomes very large for almost arbitrary inputs $\mathbf{f} \in \mathbb{C}^N$. We denote the eigenpair of smallest and second-smallest magnitude eigenvalues of $\mathbf{A}$ as $(\lambda^0, \mathbf{p}^0)$ and $(\lambda^1, \mathbf{p}^1)$. The following relationships holds according to matrix norm theory:

$$\frac{1}{|\lambda^0|} = \frac{|\mathbf{A}^{-1}\mathbf{p}^0|}{|\mathbf{p}^0|}, \quad \frac{1}{|\lambda^1|} \geq \max_{\substack{\mathbf{p} \in \mathbb{C}^N \\ \langle \mathbf{p}, \mathbf{p}^0 \rangle = 0}} \frac{|\mathbf{A}^{-1}\mathbf{p}|}{|\mathbf{p}|}. \tag{B.1}$$

Note that we assume $\mathbf{A}$ is non-Hermitian and invertible, which is typically true for exterior acoustic problems. For arbitrary input $\mathbf{f}$, the vector $\mathbf{f}_{||} = \langle \mathbf{f}, \mathbf{p}^0 \rangle \mathbf{p}^0 / |\mathbf{p}^0|^2$ is a projection of $\mathbf{f}$ in the direction of $\mathbf{p}^0$ and the vector $\mathbf{f}_{\perp} = \mathbf{f} - \mathbf{f}_{||}$ is the remaining component. The output magnitude is bounded from below as

$$|\mathbf{p}| = |\mathbf{A}^{-1}\mathbf{f}| = \left| \mathbf{A}^{-1}(\mathbf{f}_{||} + \mathbf{f}_{\perp}) \right| \tag{B.2}$$

$$\geq \left| \mathbf{A}^{-1}\mathbf{f}_{||} \right| - \left| \mathbf{A}^{-1}\mathbf{f}_{\perp} \right| \tag{B.3}$$

$$\geq \frac{|\mathbf{f}_{||}|}{|\lambda^0|} - \frac{|\mathbf{f}_{\perp}|}{|\lambda^1|}. \tag{B.4}$$

This relationship shows that the magnitude of output $|\mathbf{p}|$ will become larger than the input as long as the magnitude of λ^0 is much smaller than the magnitude of λ^1, except for the very rare case where $\mathbf{f}$ is perpendicular to $\mathbf{p}^0$.

C Sensitivity Derivation

To derive the sensitivity of the eigenvalue and resonance wavenumber (Equation (12)), we compute one iteration of the inverse power iteration in Algorithm 1 which is $\mathbf{w} = \mathbf{D}\mathbf{A}^{-1}\mathbf{v}$. Let the matrix $\mathbf{D}$ and $\mathbf{A}$ be perturbed by a geometric change as $\mathbf{D} + \epsilon\Delta\mathbf{D}$ and $\mathbf{A} + \epsilon\Delta\mathbf{A}$, where ϵ is a small number. Then $\mathbf{w}$ changes as

$$\begin{aligned}
\mathbf{w} + \epsilon\Delta\mathbf{w} &= (\mathbf{D} + \epsilon\Delta\mathbf{D})(\mathbf{A} + \epsilon\Delta\mathbf{A})^{-1}\mathbf{v}, && \text{(C.1)}\\
&= (\mathbf{D} + \epsilon\boldsymbol{\Delta}\mathbf{D})\{\mathbf{A}(\mathbf{I} + \epsilon\mathbf{A}^{-1}\Delta\mathbf{A})\}^{-1}\mathbf{v}, && \text{(C.2)}\\
&= (\mathbf{D} + \epsilon\Delta\mathbf{D})(\mathbf{I} + \epsilon\mathbf{A}^{-1}\Delta\mathbf{A})^{-1}\mathbf{A}^{-1}\mathbf{v}, && \text{(C.3)}\\
&\simeq (\mathbf{D} + \epsilon\Delta\mathbf{D})(\mathbf{I} - \epsilon\mathbf{A}^{-1}\Delta\mathbf{A})\mathbf{x}, && \text{(C.4)}\\
&= \underbrace{\mathbf{D}\mathbf{A}^{-1}\mathbf{v}}_{=\mathbf{w}} + \epsilon\underbrace{\left(\Delta\mathbf{D}\mathbf{x} - \mathbf{D}\mathbf{A}^{-1}\Delta\mathbf{A}\mathbf{x}\right)}_{=\Delta\mathbf{w}}. && \text{(C.5)}
\end{aligned}$$

We use the Neuman expansion for the transformation from (C.3) to (C.4). We ignore the term $\Delta\mathbf{D}\mathbf{x}$ since we observed that its contribution is very small.

References

[1] Burton, A., and Miller, G. The application of integral equation methods to the numerical solution of some exterior boundary-value problems. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences* (1971), vol. 323, The Royal Society, pp. 201–210.

[2] Kirkup, S. M. *The Boundary Element Method in Acoustics: A Development in Fortran (Integral Equation Methods in Engineering).* Integrated Sound Software, 10 1998.

[3] Van Oosterom, A., and Strackee, J. The solid angle of a plane triangle. *IEEE Transactions on Biomedical Engineering 2*, BME-30 (1983), 125–126.