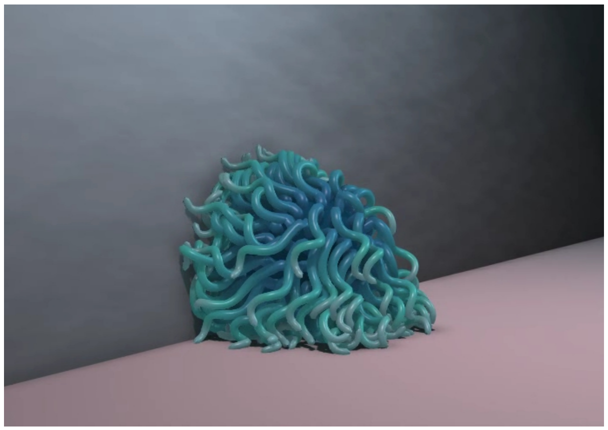
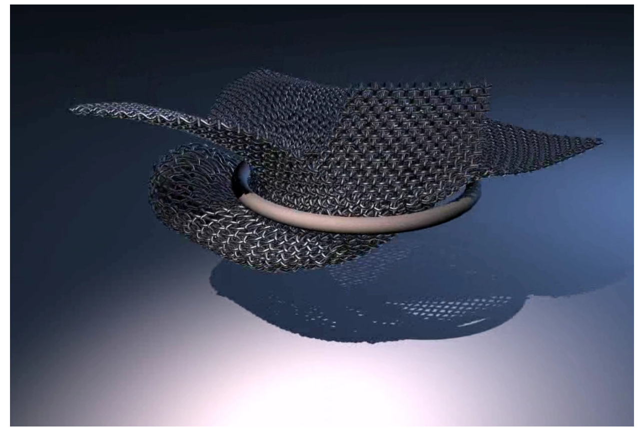
Position-based Elastic Rods





Nobuyuki Umetani
Ryan Schmidt
Jos Stam

AUTODESK
RESEARCH

Motivation

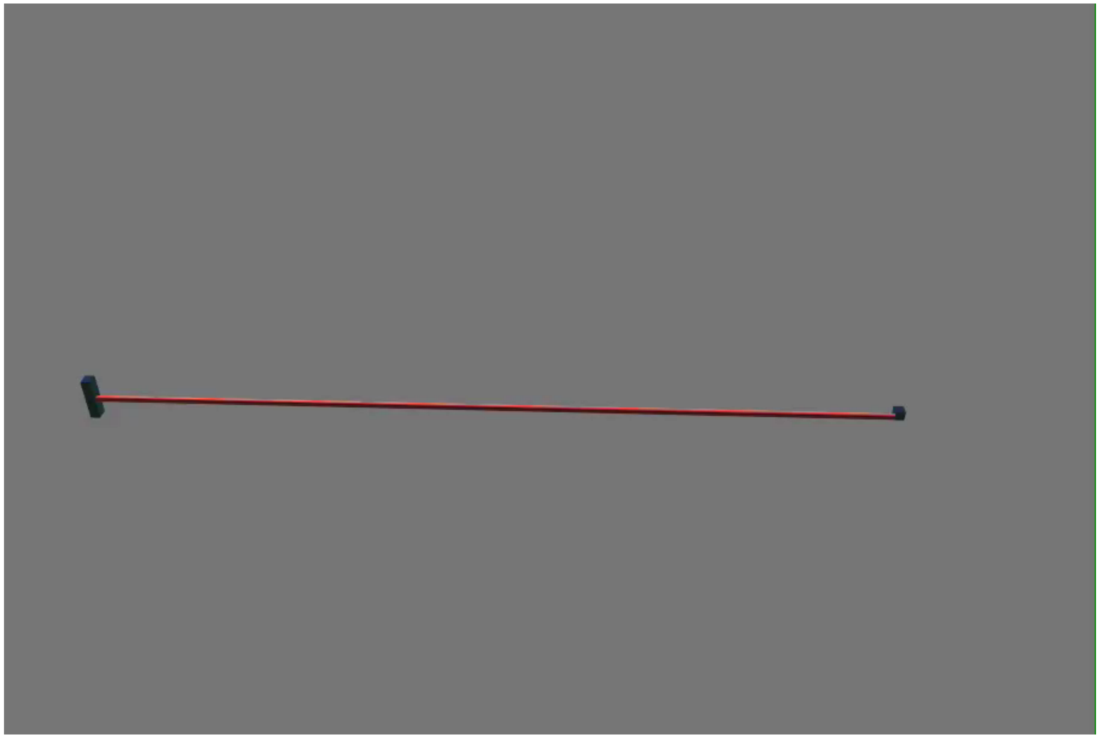
- Rods are everywhere






Complexity of Rod's Behavior

- Rich nonlinear deformation arising from coupling of bending and twisting



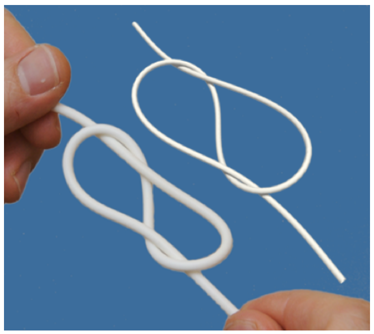
Goal

- Fast, robust, simple implementation
- Qualitatively good results
- Coupling with other physics models

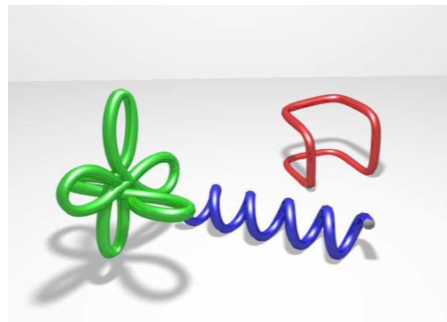
Position-based Dynamics (PBD)

Elastic Rod Simulations

- All previous works are ***not*** position-based approach



[Bergou et al. 2008]

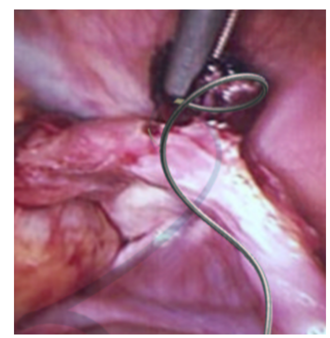


[Spillmann et al. 2007]

[Casati et al. 2013]

[Bergou et al. 2010]

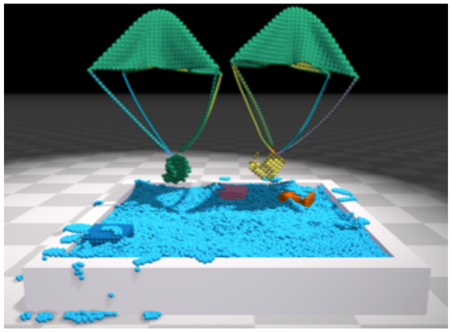
[Bartails et al. 2006]



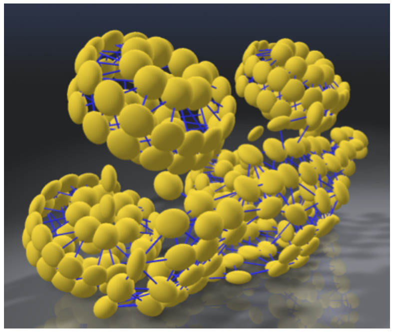
[Pai 2002]

Position-based Dynamics (PBD)

- Elastic rods have yet not been solved using PBD



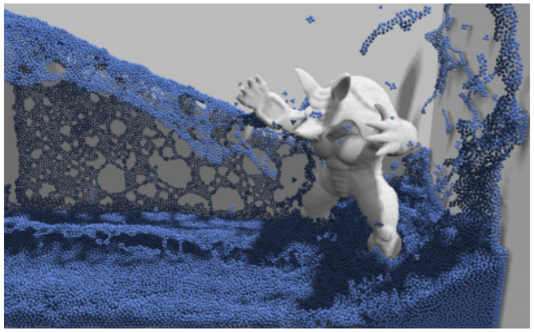
[Macklin et al. 2014]



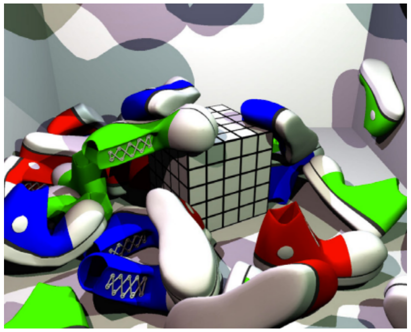
[Müller et al. 2011]

[Stam 2009]

[Müller et al. 2006]



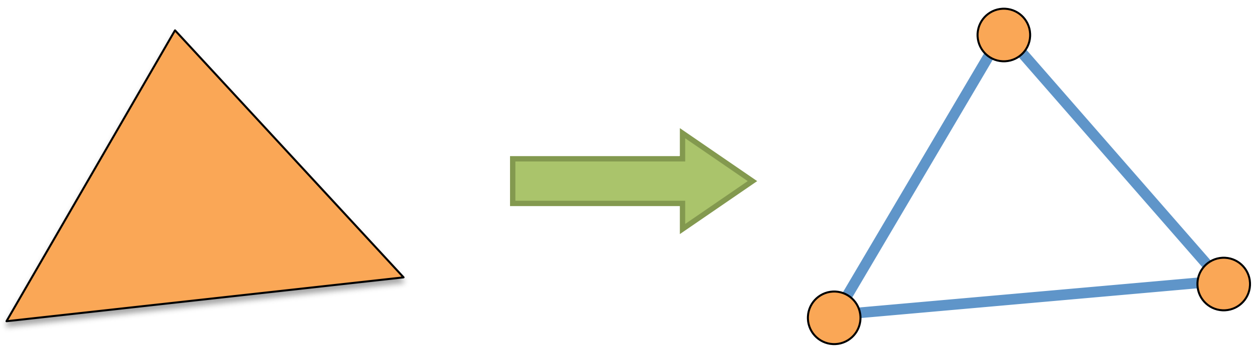
[Macklin et al. 2013]



[Müller et al. 2005]

What is PBD ?: Representation

- Shape is represented in ***positions***

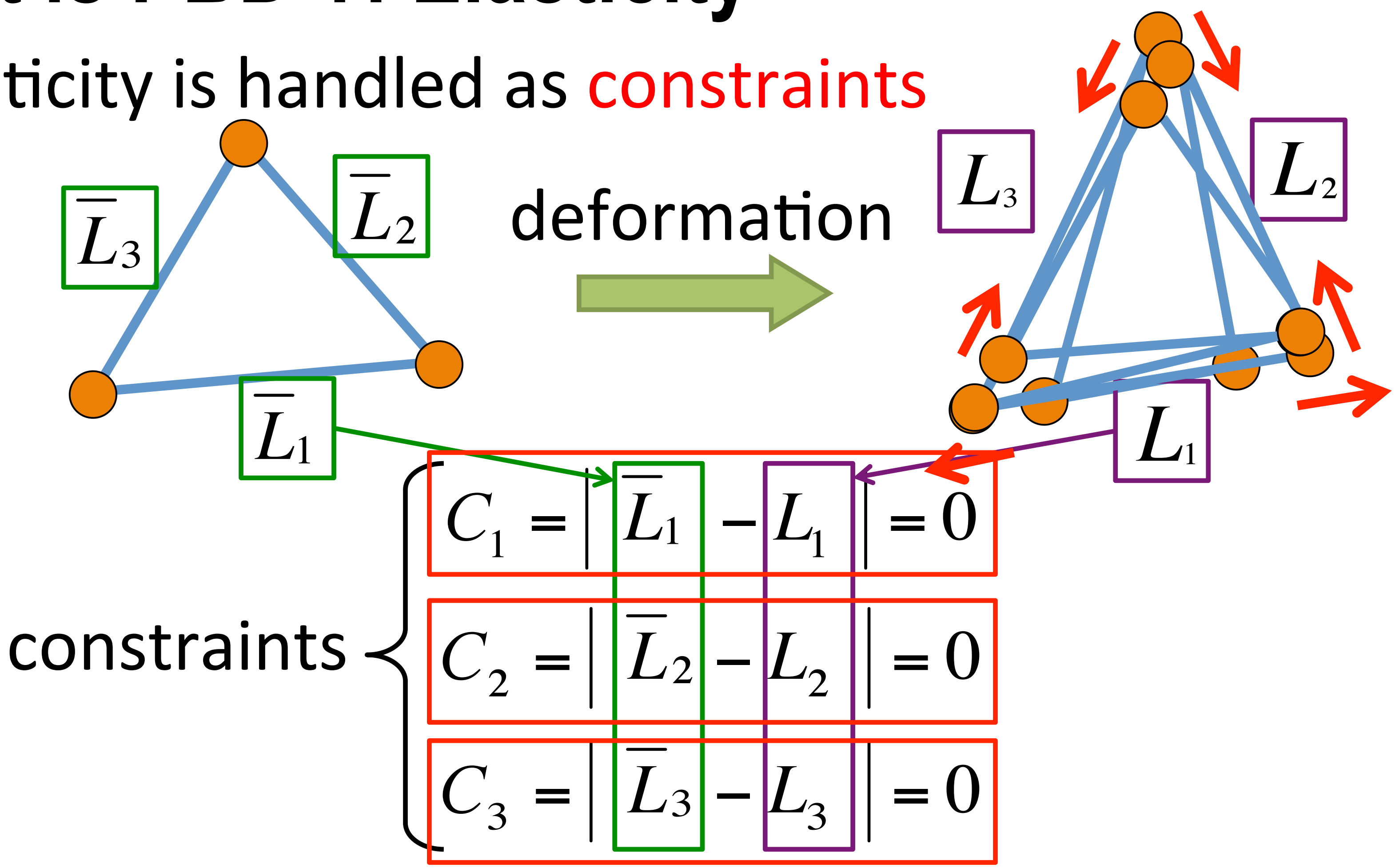


Animate with
Verlet time integration

$$\begin{cases} \mathbf{p}^{t+\Delta t} = \mathbf{p}^{t} + \Delta t(\mathbf{v}^{t} + \Delta t\mathbf{g}) + \Delta\mathbf{p} \\ \mathbf{v}^{t+\Delta t} = (\mathbf{p}^{t+\Delta t} - \mathbf{p}^{t}) / \Delta t \end{cases}$$

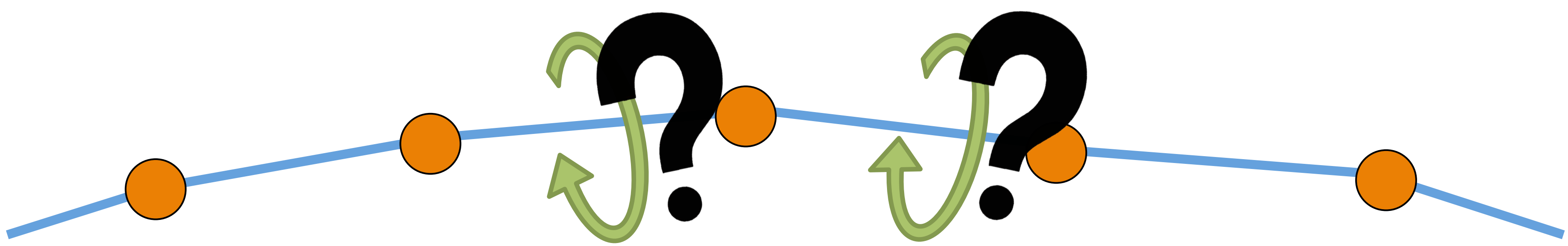
What is PBD ?: Elasticity

- Elasticity is handled as constraints

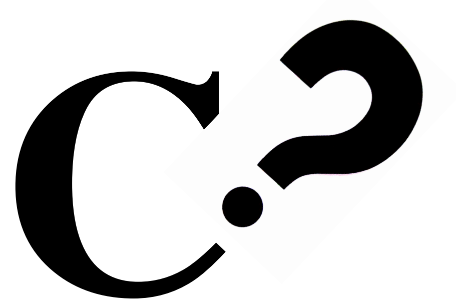


Why Rods are Difficult for PBD?

- Twist is difficult to represent with positions



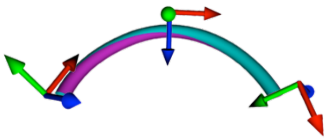
- What is the constraint, coupling bend and twist?



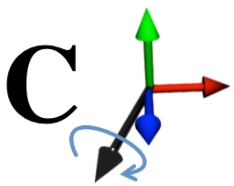
Contributions

- Twist representation for the position-based rod
  - Darboux vector to represent twist and bending
  - Ghost point for material frame definition
- Constraints for twist-bend coupling
  - Cossart theory based formulation
- Handling of vector type constraint
  - Variational formulation

Outline

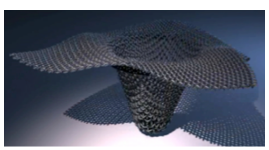


Twist representation for PDB



Constraints for twist and bending

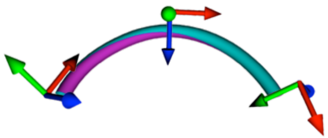
Handling vector type constraint



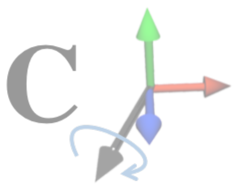
Result

Discussion

Outline

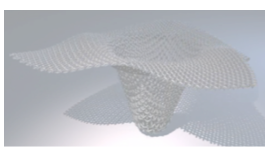


Twist representation for PDB



Constraints for twist and bending

Handling vector type constraint

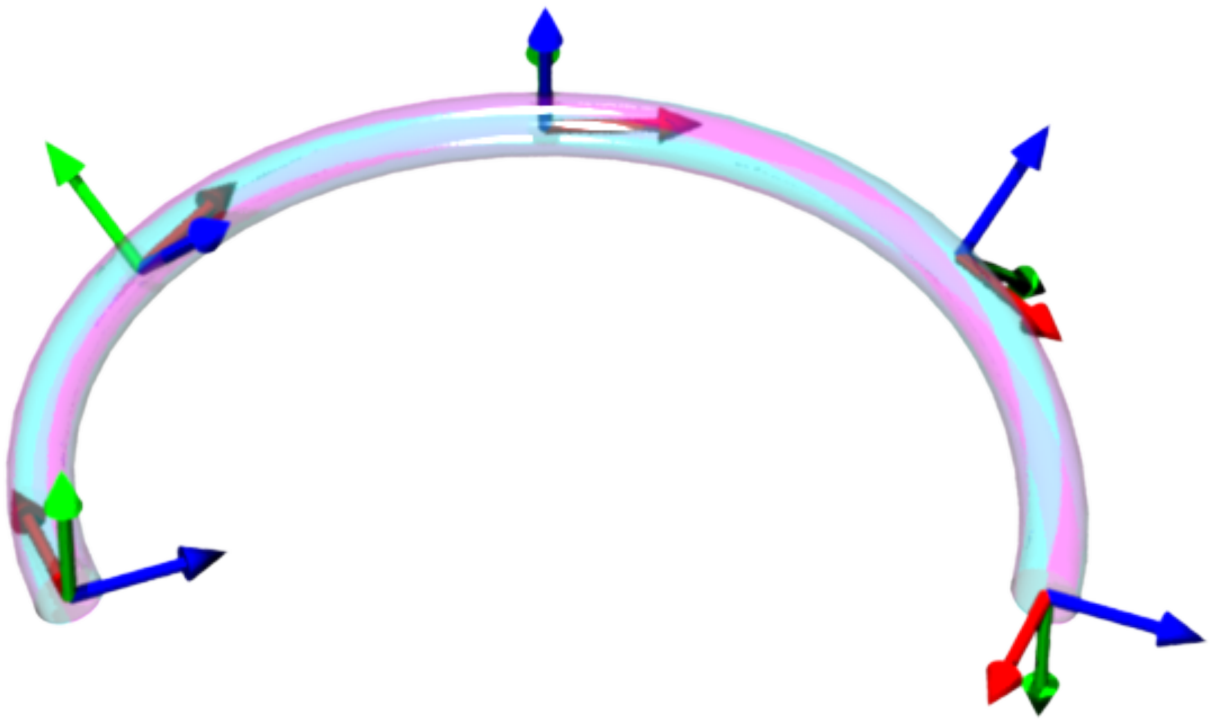


Result

Discussion

Material Frame on a Rod

- Adaptive frame: the rod's tangent = a frame axis

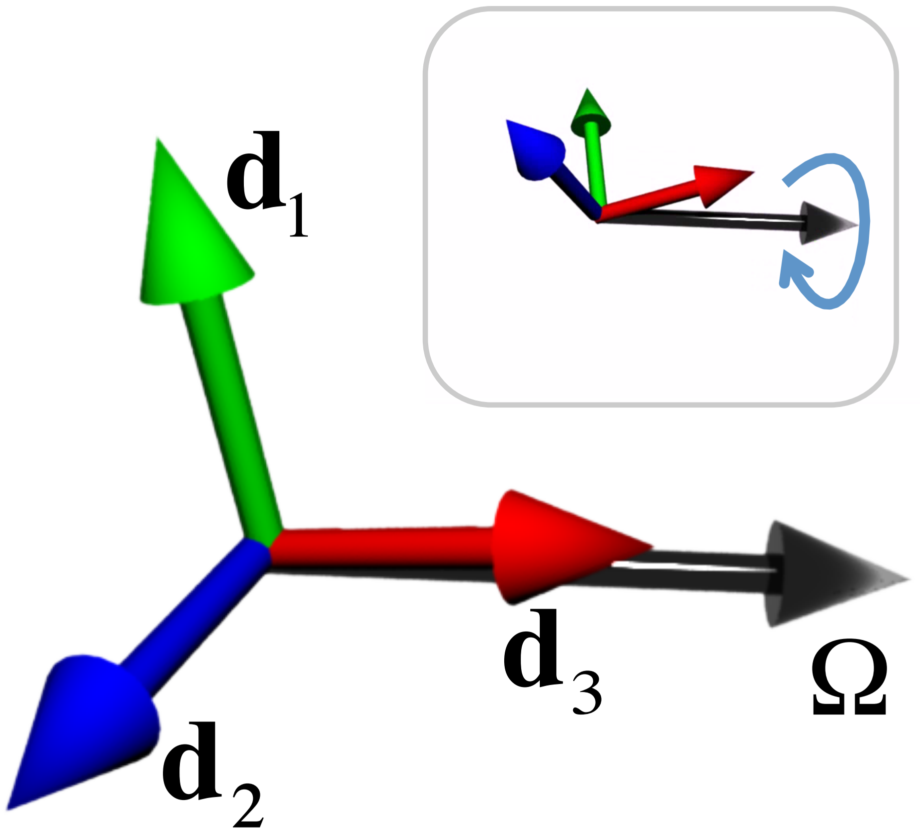


Material Frame Rotation along Rod's Axis

- Frame is rotating along the black axis

Bending & Twisting as the Frame Rotation

- Coordinate rotation axis is Darboux vector



$$\Omega_3 = \Omega \cdot \mathbf{d}_3$$

twisting

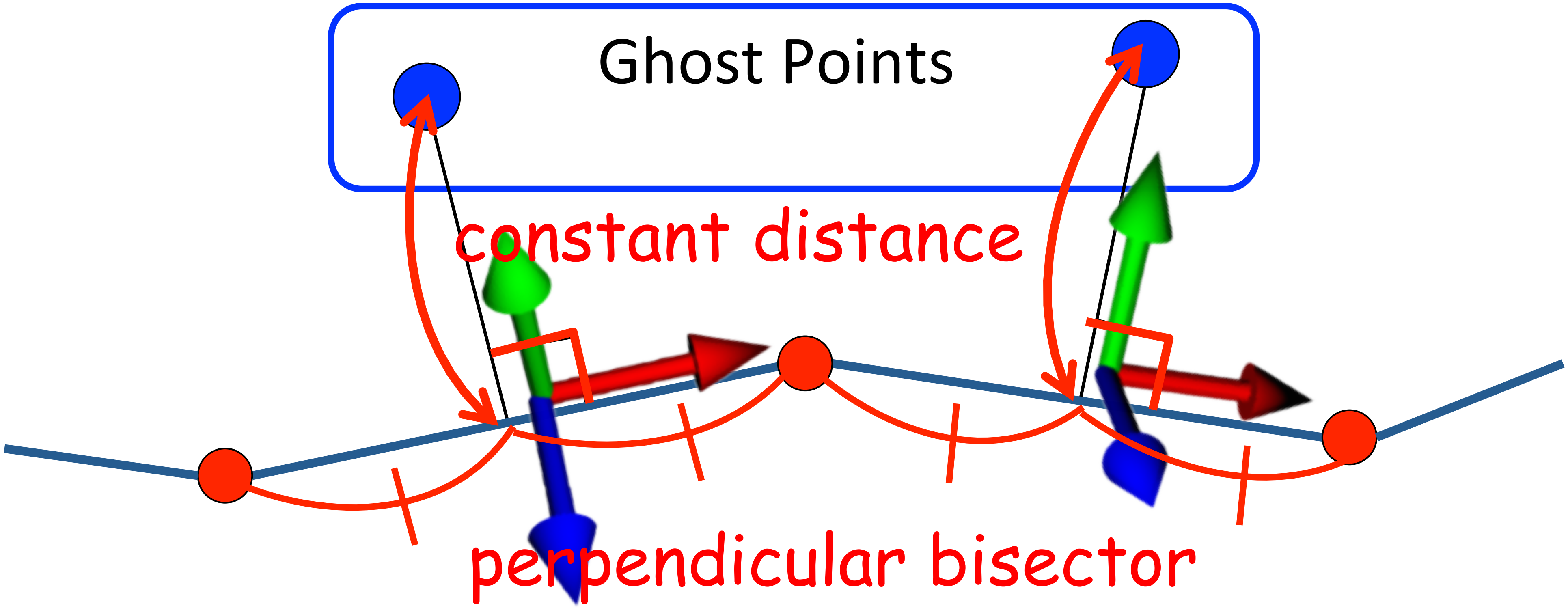
$$\Omega_1 = \Omega \cdot \mathbf{d}_1$$

$$\Omega_2 = \Omega \cdot \mathbf{d}_2$$

bending

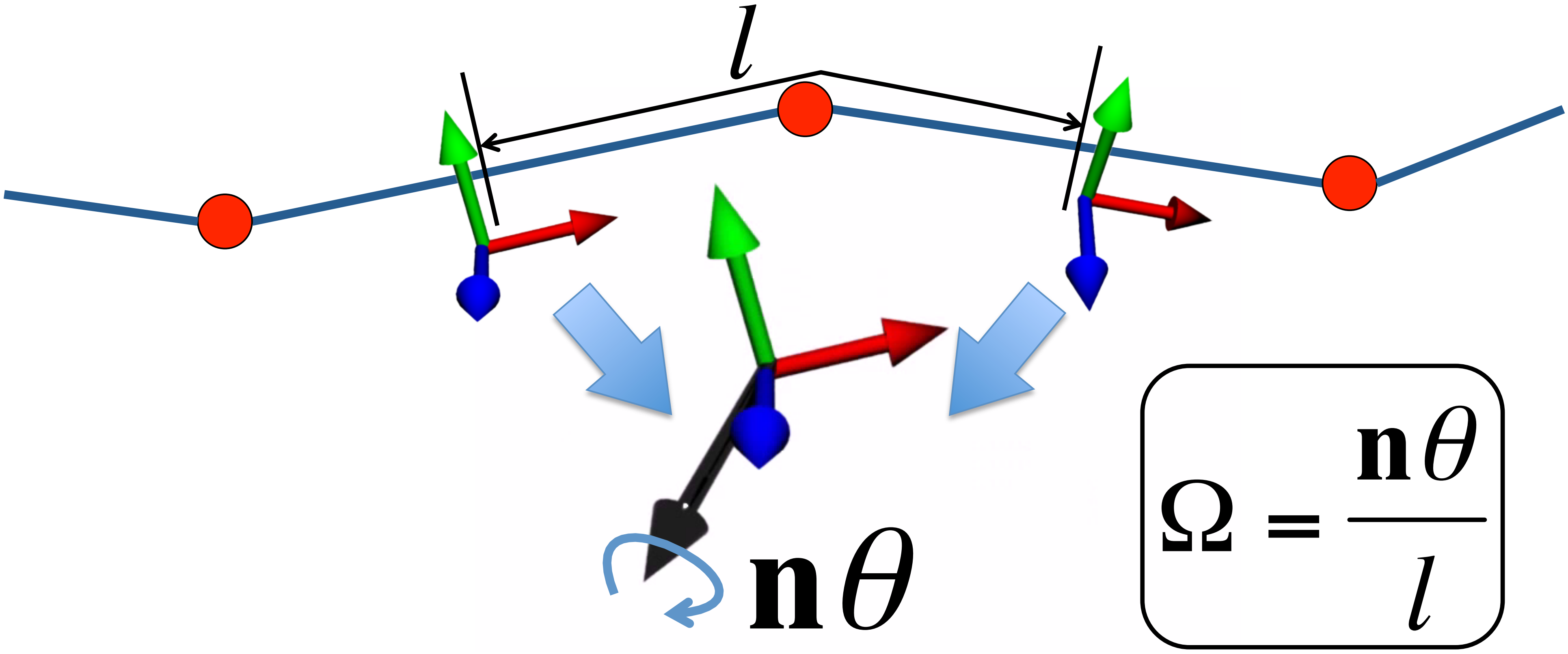
Ghost Points for Frame Representation

- Ghost point is assigned at the center of edge



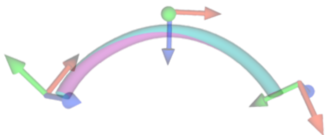
Frame Rotation in Discrete Edges

- Constant speed rotation between edges

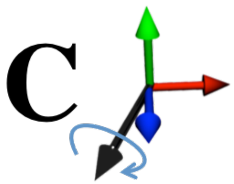


$$\Omega = \frac{\mathbf{n}\theta}{l}$$

Outline

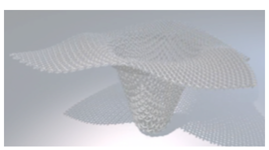


Twist representation for PDB



Constraints for twist and bending

Handling vector type constraint



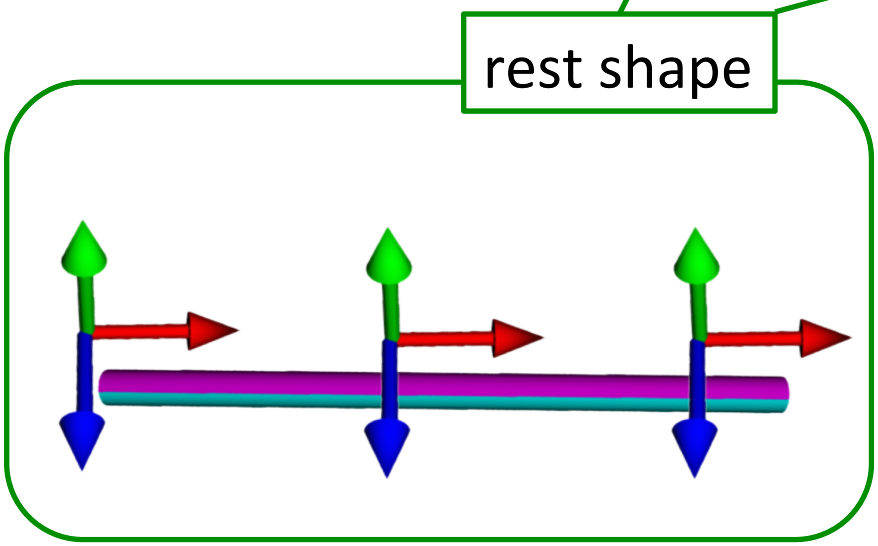
Result

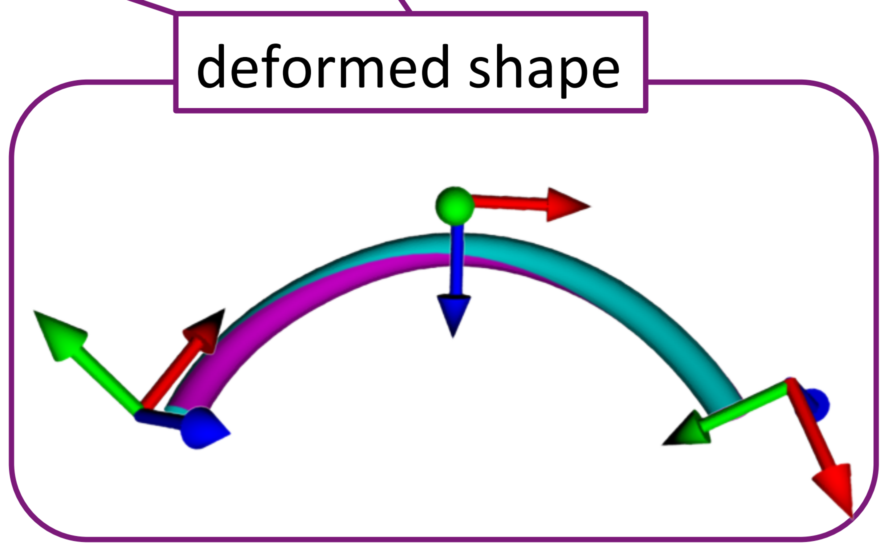
Discussion

Continuous Mechanics: Cosserat Theory

- Strain energy is Darboux vector difference

$$E = \sum_{i=1}^{3}\sum_{j=1}^{3}\left(\overline{\Omega}_i - \Omega_i\right)K_{ij}\left(\overline{\Omega}_j - \Omega_j\right)$$





Constraints for Twist & Bending

- Constraint: Darboux vector do not change

rest shape

deformed shape

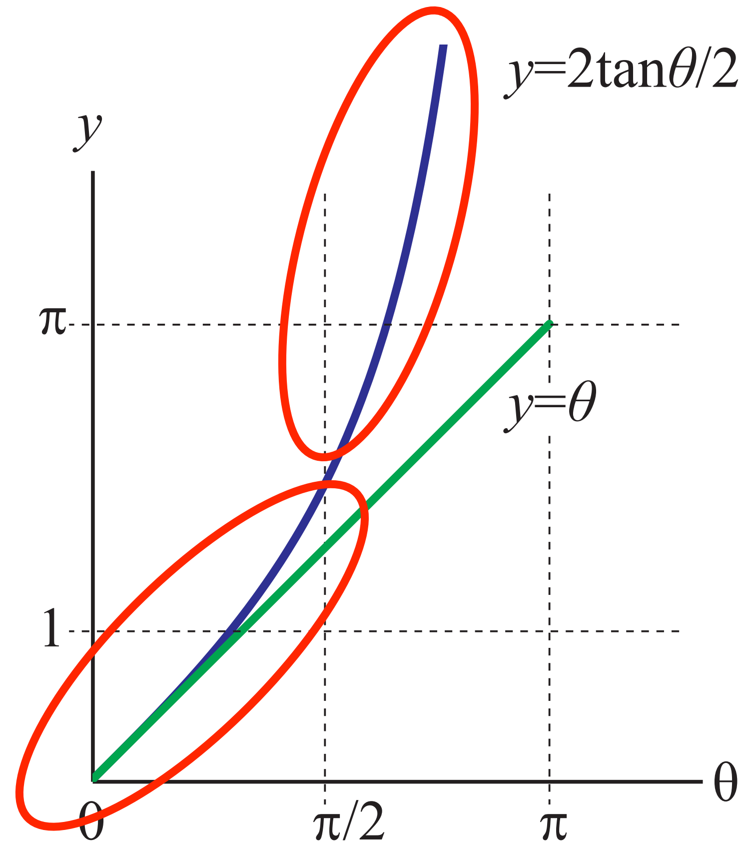
$$\mathbf{C} = \begin{pmatrix} \overline{\Omega}_1 \\ \overline{\Omega}_2 \\ \overline{\Omega}_3 \end{pmatrix} - \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} = 0$$

Modification on Discrete Darboux Vector

modification

$$\boldsymbol{\Omega} = \mathbf{n}\theta \cong \mathbf{n}2\tan\frac{\theta}{2}$$

- Good approximation at $\theta << 1$
- Prevent flipping at $\theta \approx \pi$

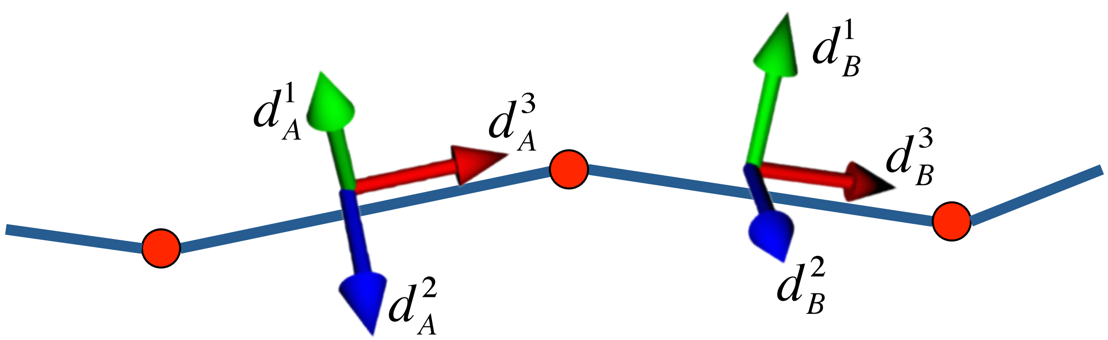


Modified Discrete Darboux Vector

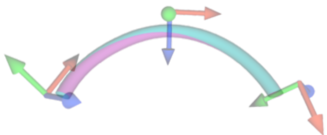
- Simple formula without trigonometry

$$\Omega_i = \left(\frac{2}{l}\right)\frac{\mathbf{d}_A^j \cdot \mathbf{d}_B^k - \mathbf{d}_A^k \cdot \mathbf{d}_B^j}{1 + \sum_{n=1}^{3} \mathbf{d}_A^n \cdot \mathbf{d}_B^n}$$

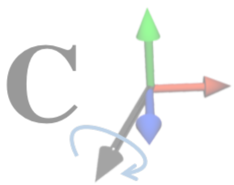
$$\{i, j, k\} = \{1,2,3\}, \{2,3,1,\}, \{3,1,2\}$$



Outline

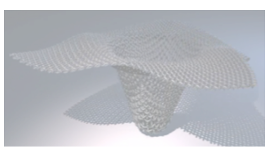


Twist representation for PDB



Constraints for twist and bending

Handling vector type constraint



Result

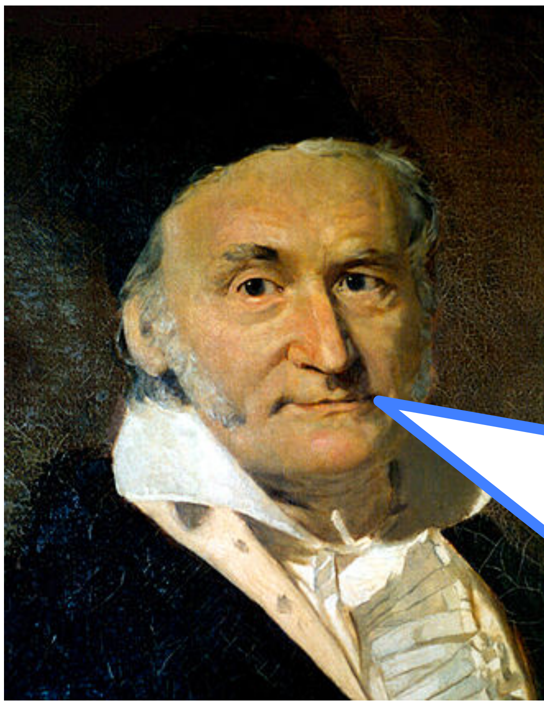
Discussion

Vector Type Constraint: Naïve Approaches

$$\mathbf{C} \in \mathbb{R}^3$$

- One-by-one approach $C' = \mathbf{C}_i = 0$
  - Slow convergence requires many steps ☹
- Energy-based approach $C' = \|\mathbf{C}\| = 0$
  - Slow convergence due to nonlinearity ☹

Gauss's Principle of Least Constraint



[Carl Friedrich Gauss 1829]

Constrained motion minimize

Acceleration change due to constraint

$$Z = \sum_i m_i \left| \ddot{\mathbf{p}}_i - \mathbf{g} \right|^2$$

mass

gravity

acceleration

Gauss's Principle in Discrete Setting

$$Z = \sum_i m_i \left| \ddot{\mathbf{p}}_i - \mathbf{g} \right|^2$$

continuous representation

Constraint update

Verlet integration

$$\mathbf{p}^{t+\Delta t} = \mathbf{p}^t + \Delta t(\mathbf{v}^t + \Delta t\mathbf{g}) + \boxed{\Delta \mathbf{p}}$$

$$\mathbf{v}^{t+\Delta t} = (\mathbf{p}^{t+\Delta t} - \mathbf{p}^t) / \Delta t$$

$$\ddot{\mathbf{p}} = (\mathbf{v}^{t+\Delta t} - \mathbf{v}^t) / \Delta t = \Delta \mathbf{p} / \Delta t^2 + \mathbf{g}$$

$$Z = \sum_i m_i \left| \Delta \mathbf{p}_i \right|^2$$

Position based representation

Variational Interpretation of Constraint Update

$$\Delta\mathbf{p} = \arg\min_{\Delta p} \sum_{s\in S} m_s \left|\Delta\mathbf{p}_s\right|^2, \quad \textit{where} \quad \mathbf{C}(\mathbf{p}+\Delta\mathbf{p}) = 0$$

QP problem

linearized constraint

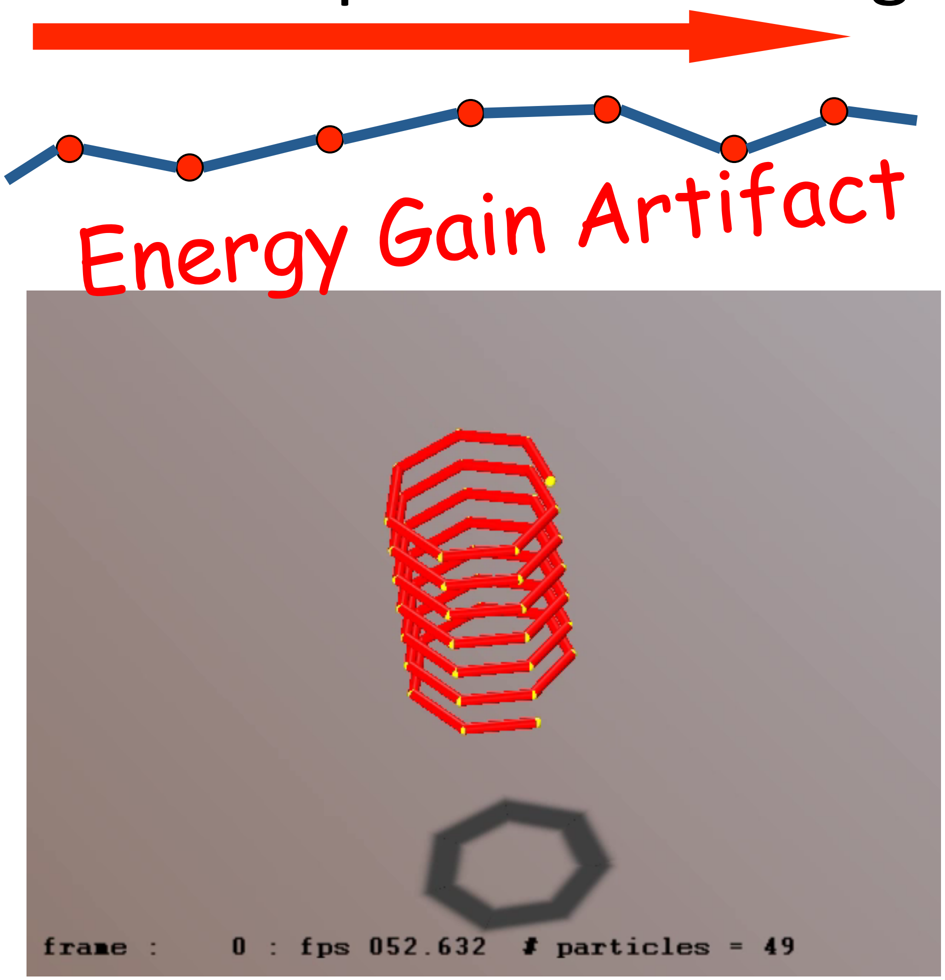
Block Gauss-Seidel

$$\Delta\mathbf{p}_s = -\frac{1}{m_s}\nabla_s\mathbf{C}^T\left(\sum_{t\in S}\frac{1}{m_t}\nabla_t\mathbf{C}^T\nabla_t\mathbf{C}\right)^{-1}\mathbf{C}$$

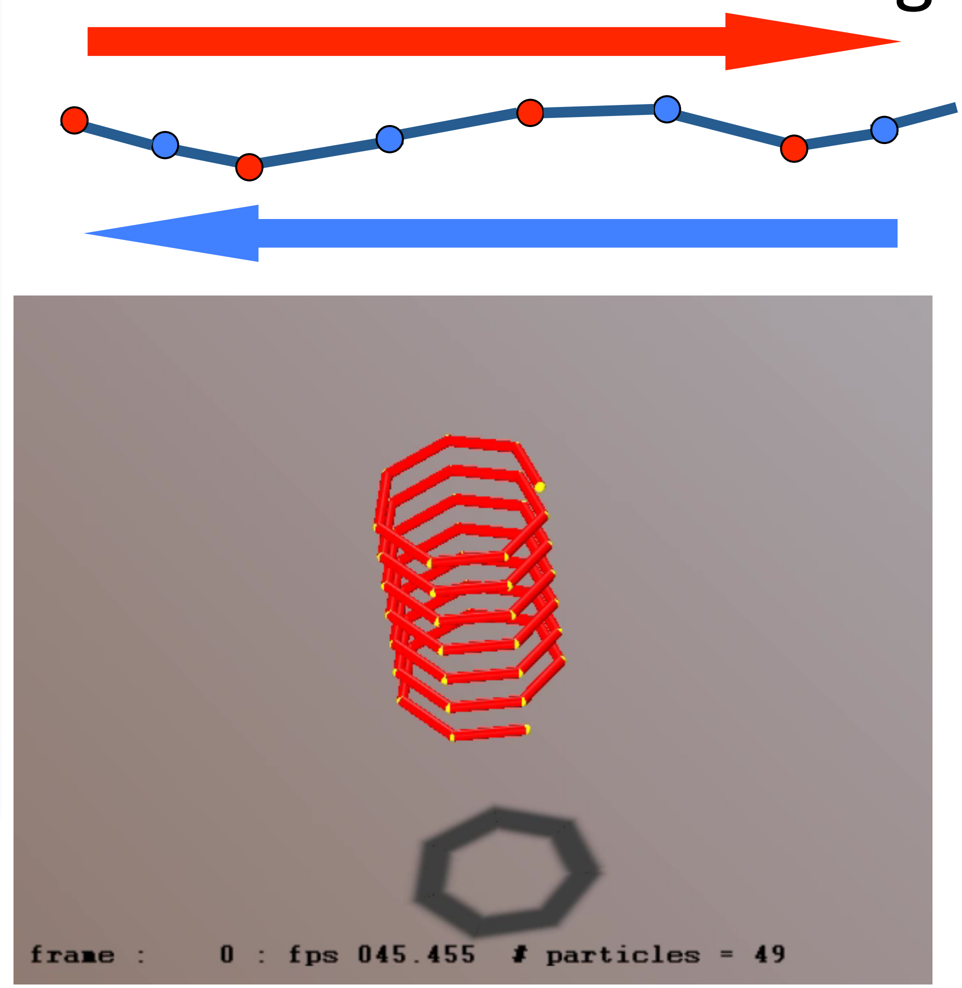
final update formula

Order Dependent Stability

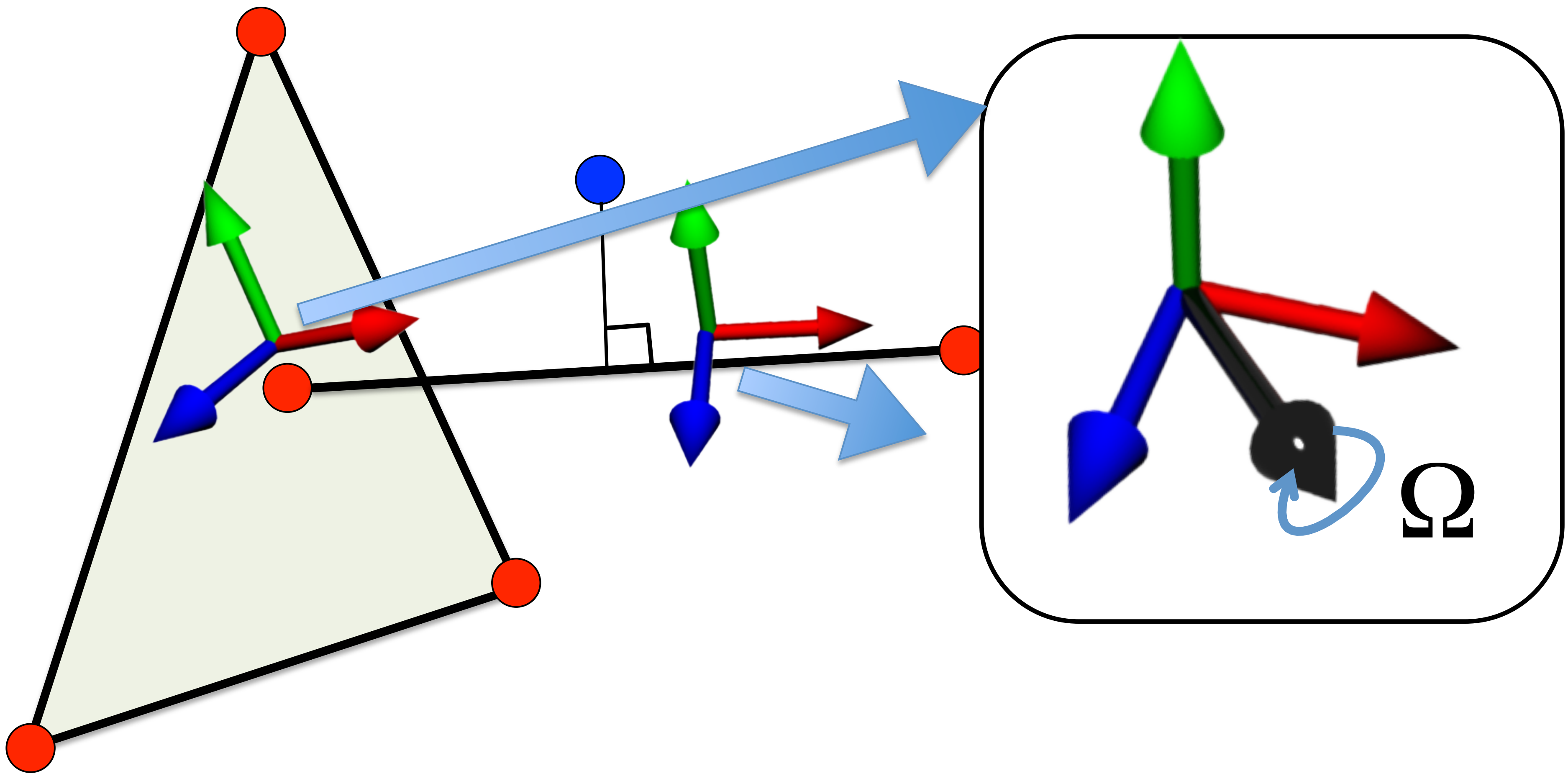
Naïve sequential ordering



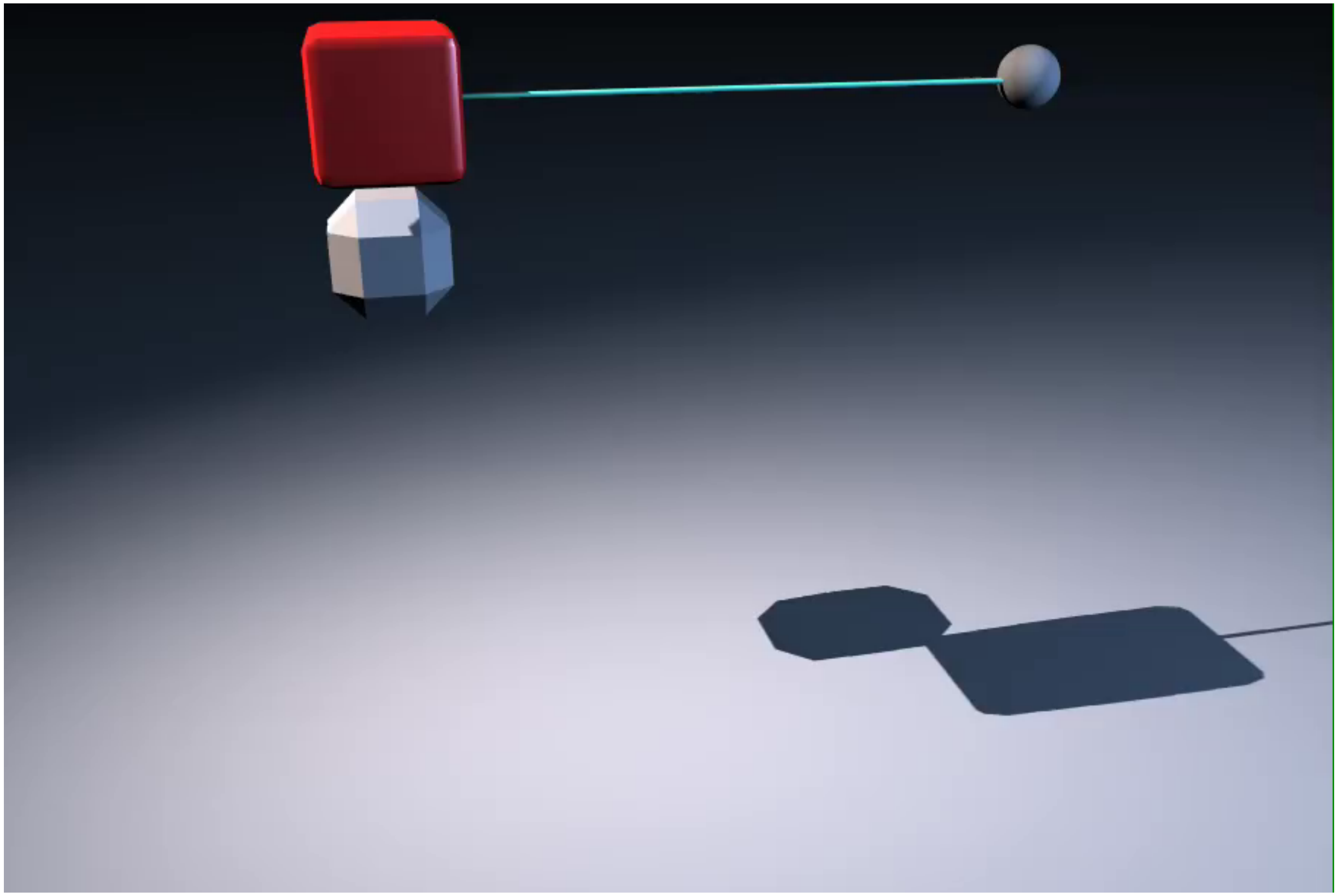
Our bidirectional ordering



Coupling with Triangle at End Point

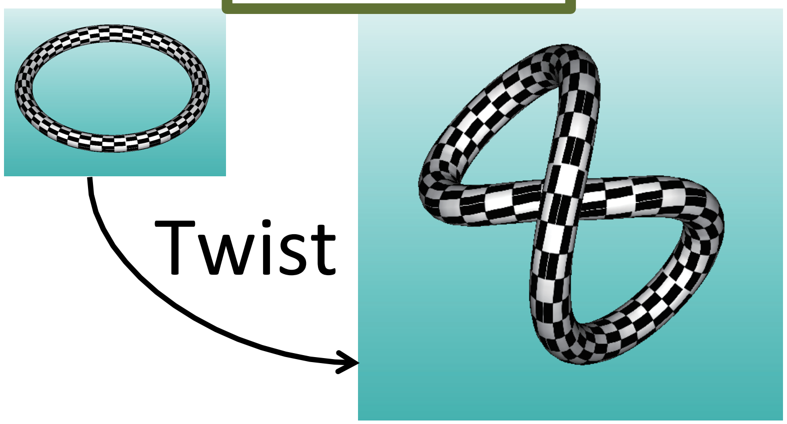


Coupling Rod and Triangle Mesh



Anisotropic Stiffness Adjustment

isotropic



$$\mathbf{C} = \begin{pmatrix} \overline{\Omega}_1 \\ \overline{\Omega}_2 \\ \overline{\Omega}_3 \end{pmatrix} - \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} = 0$$

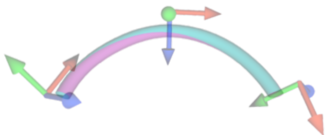
anisotropic

$$\Omega_i^{goal} = \alpha_i \overline{\Omega}_i + (1 - \alpha_i)\Omega_i$$

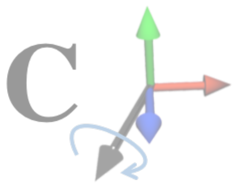
$$\alpha_i : stiffness$$

$$\mathbf{C} = \begin{pmatrix} \Omega_1^{goal} \\ \Omega_2^{goal} \\ \Omega_3^{goal} \end{pmatrix} - \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} = 0$$

Outline

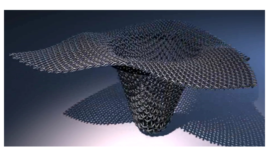


Twist representation for PDB



Constraints for twist and bending

Handling vector type constraint

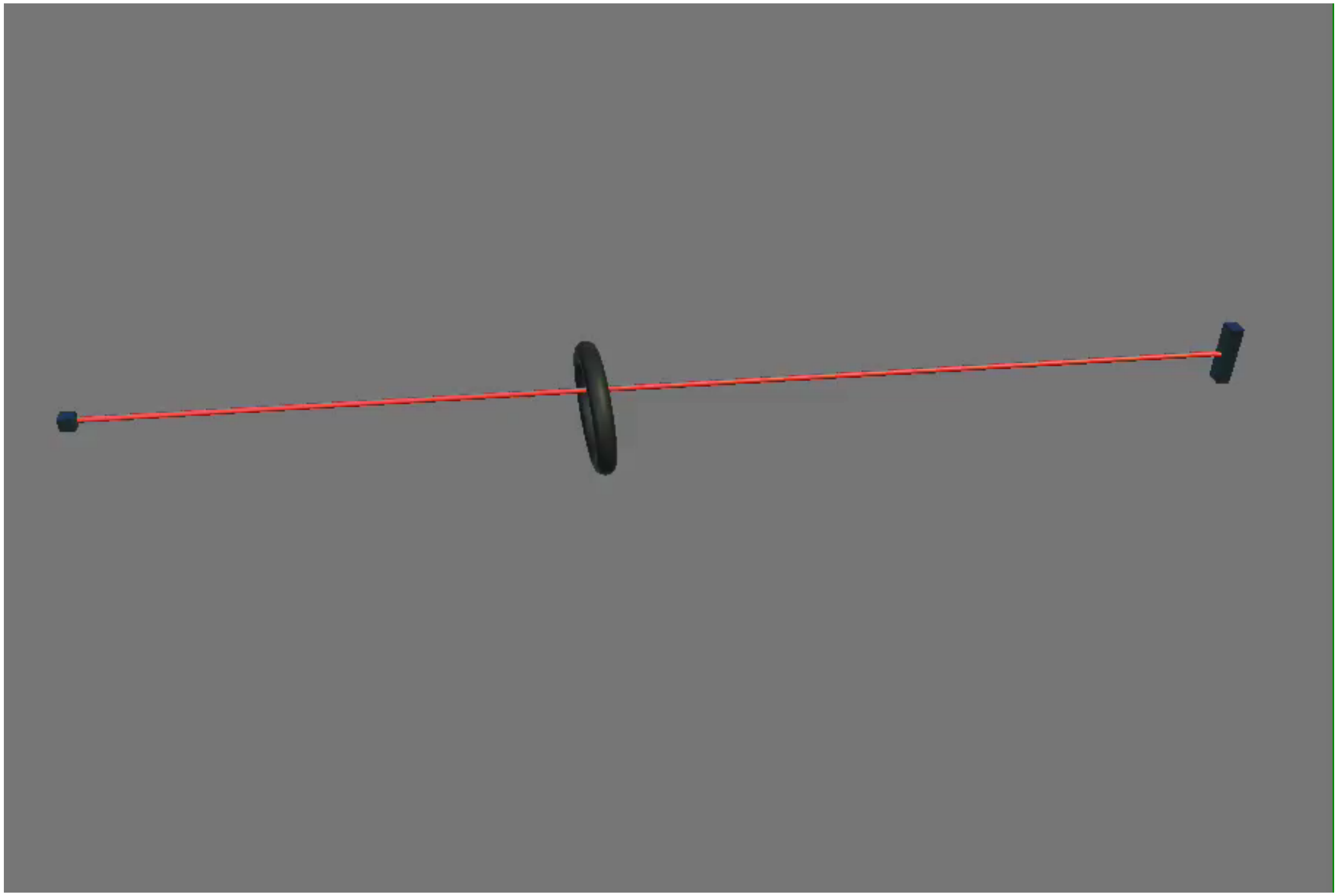


Result

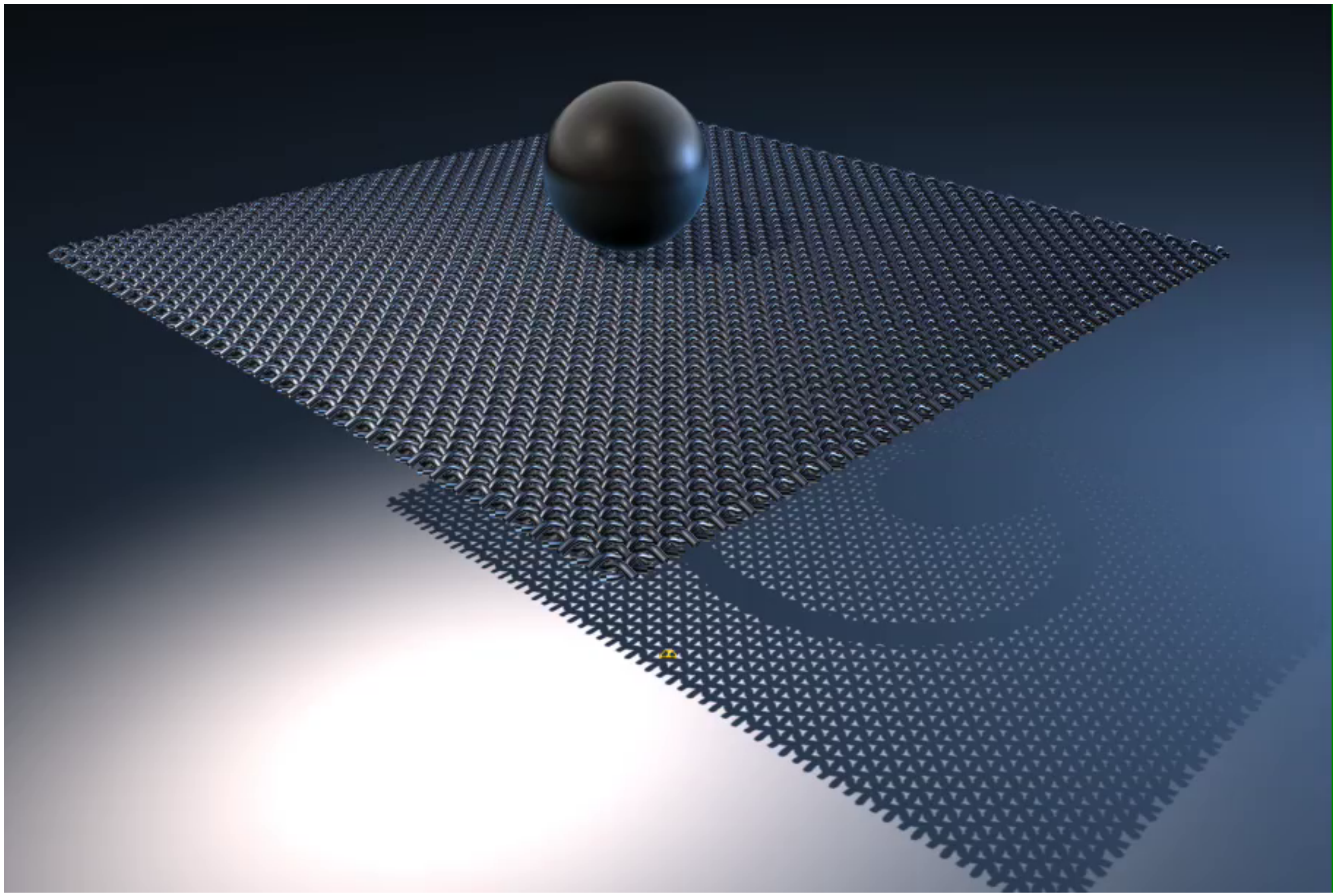
Discussion

Result: Stability

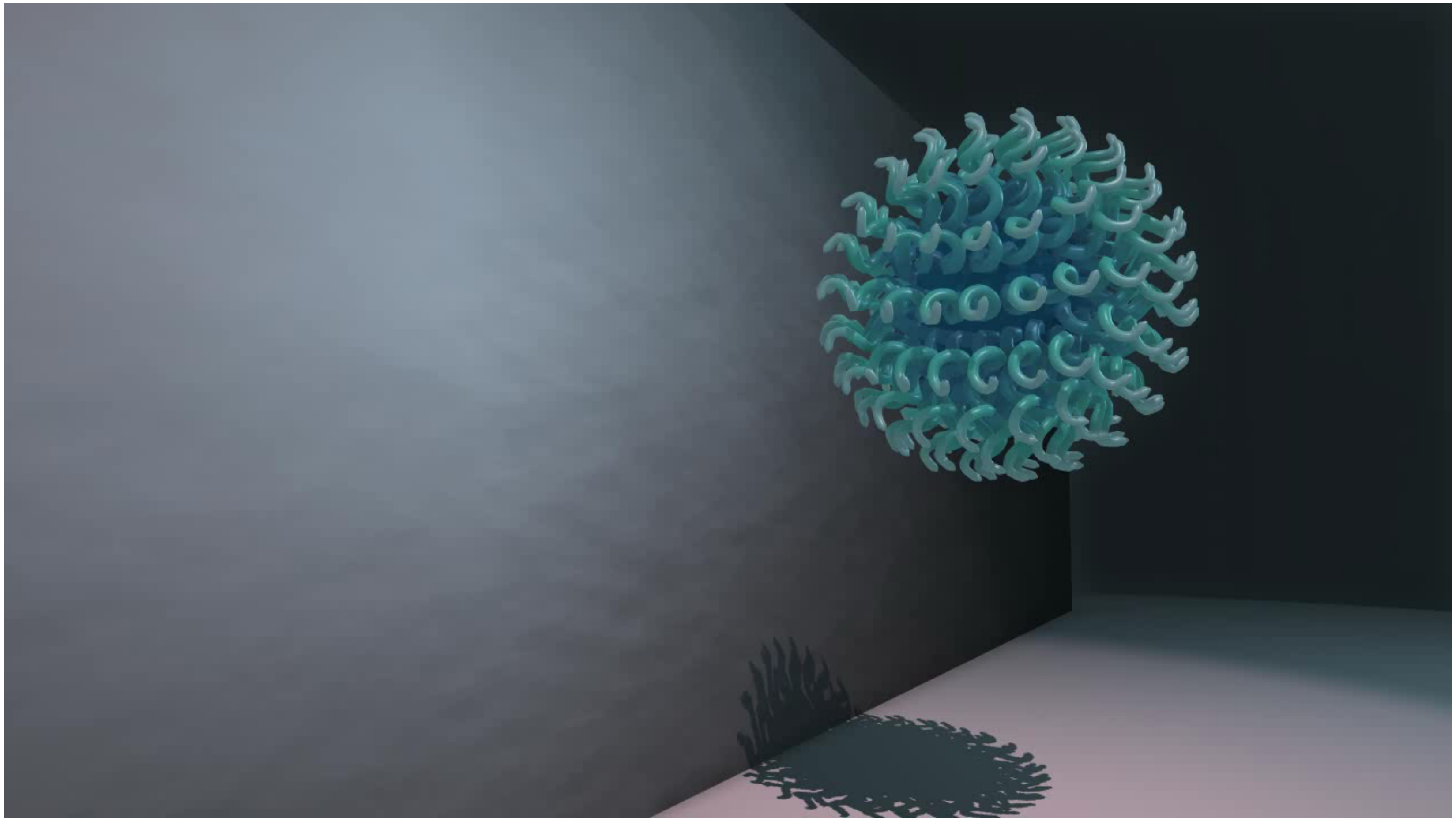
Result: Large Twist



Result: Wire Mesh



Result: Koosh Ball



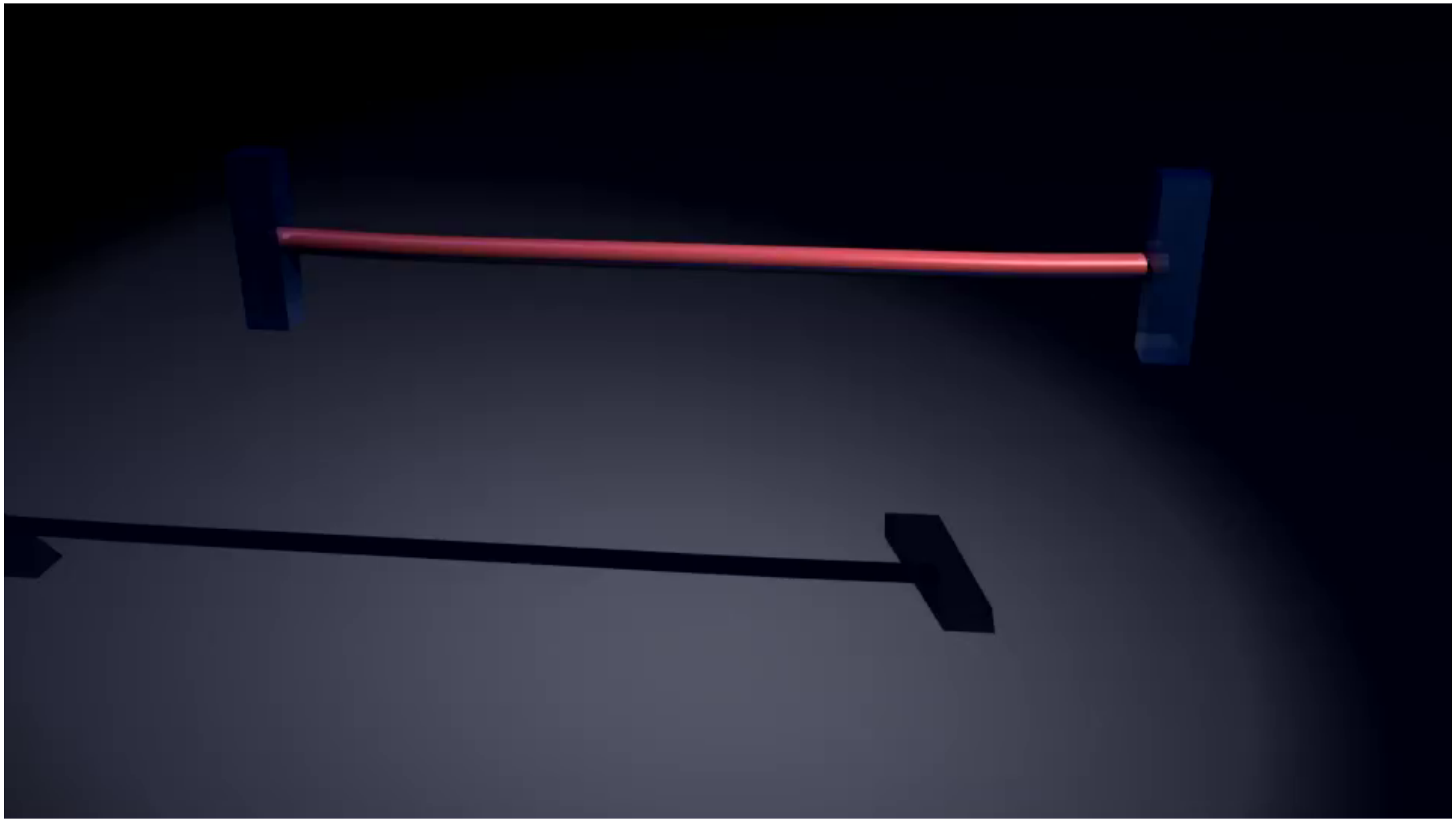
Comparison

- Our technique
- Shape matching + twisting force [Rungiratananon et al. 2012]
- Oriented particles [Müller et al. 2011]

- Same discretization and iteration number
- No self-intersection

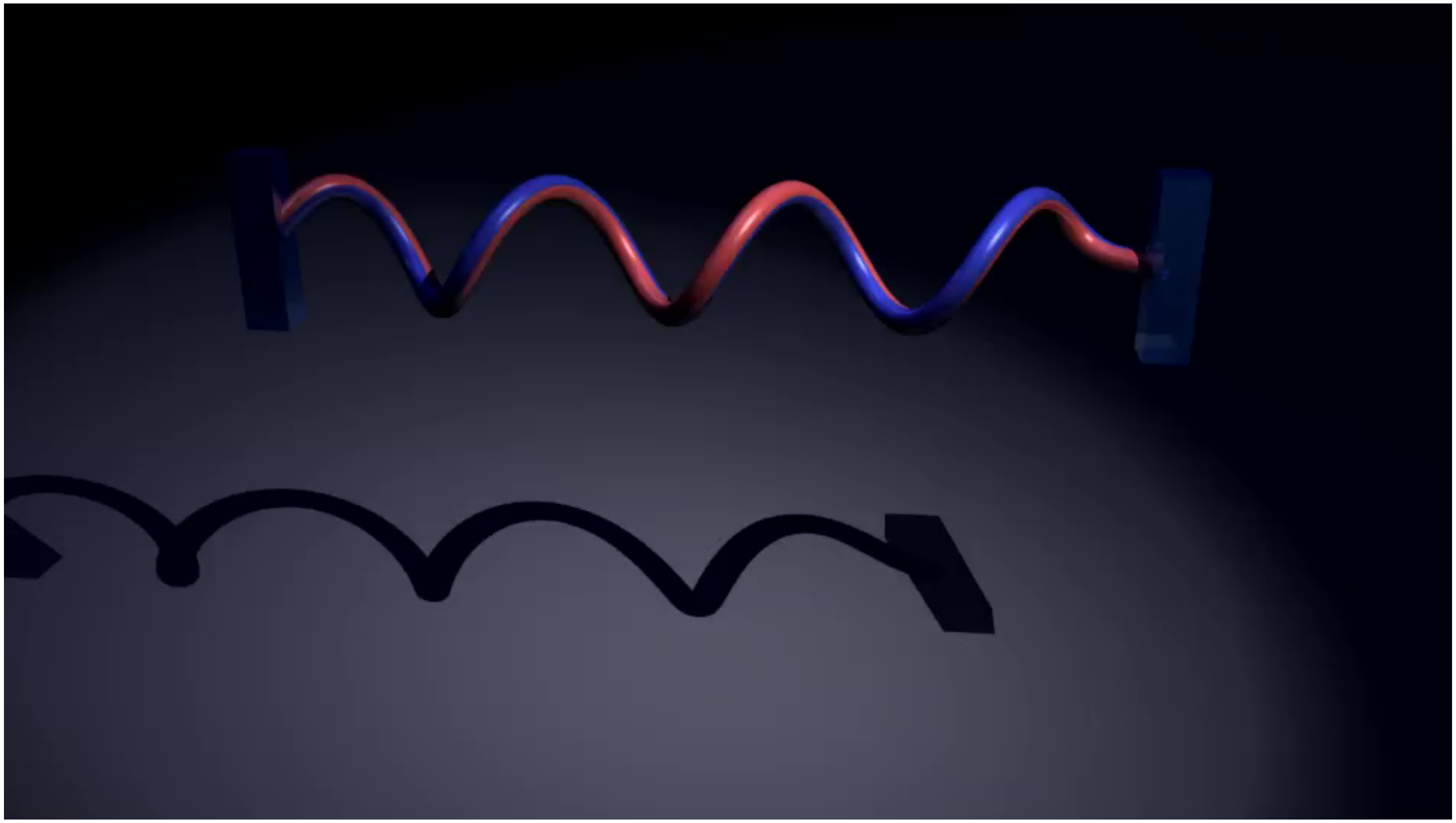
Our Technique

- Stable coupling twist and bending



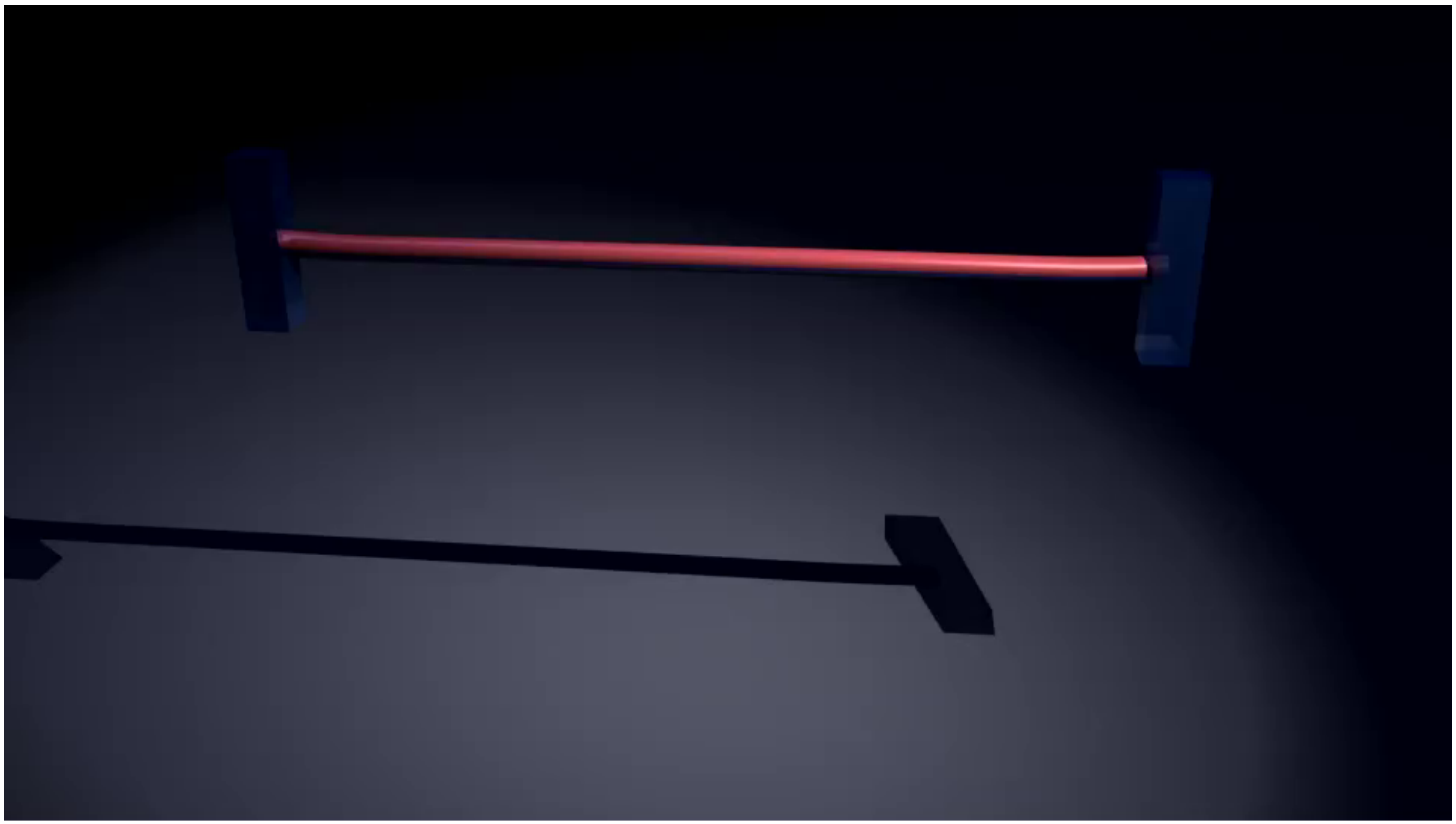
Our Technique

- Stable coupling twist and bending



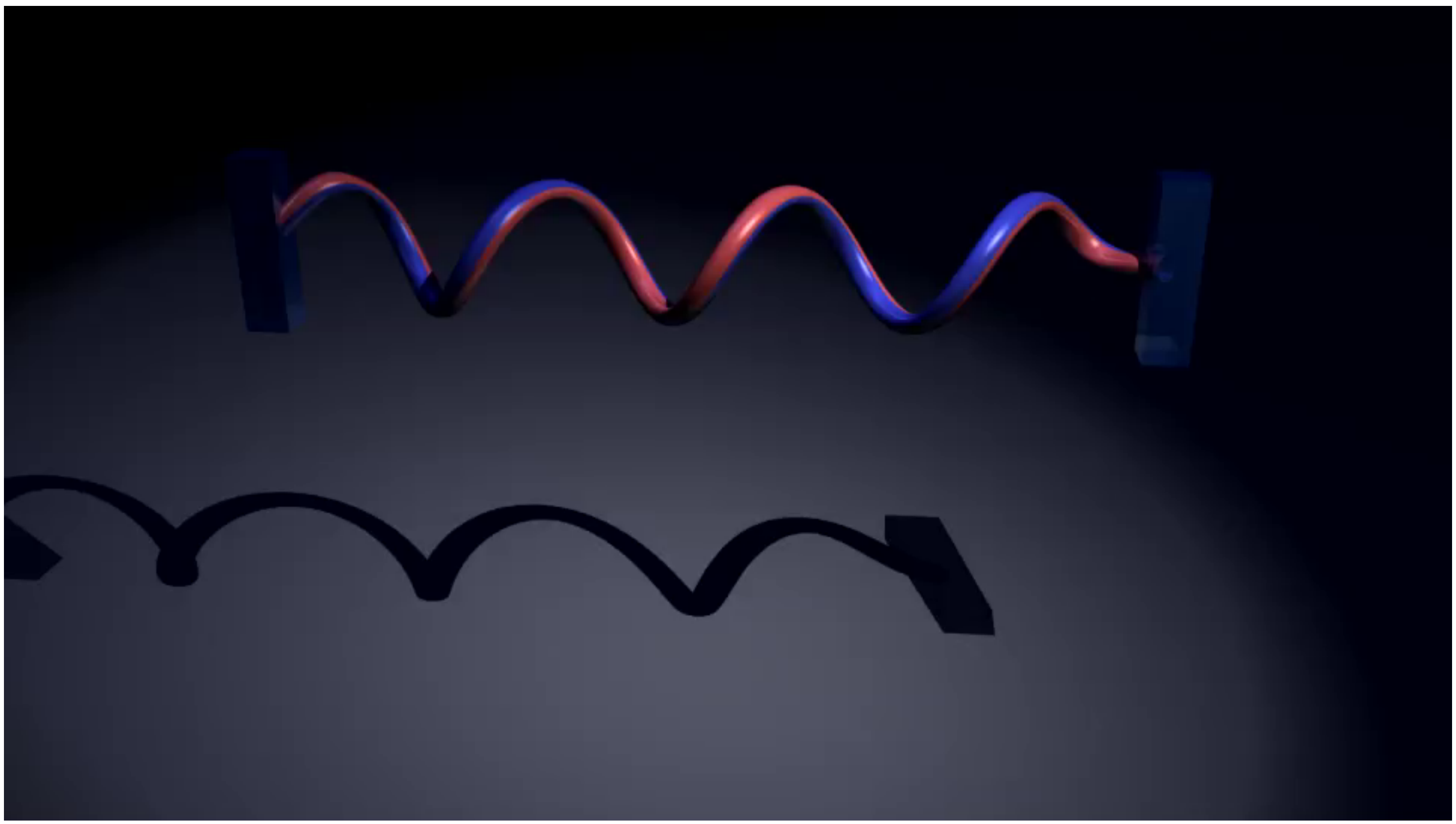
Shape Matching+Twisting Force [Rungiratananon et al. 2012]

- No convergence



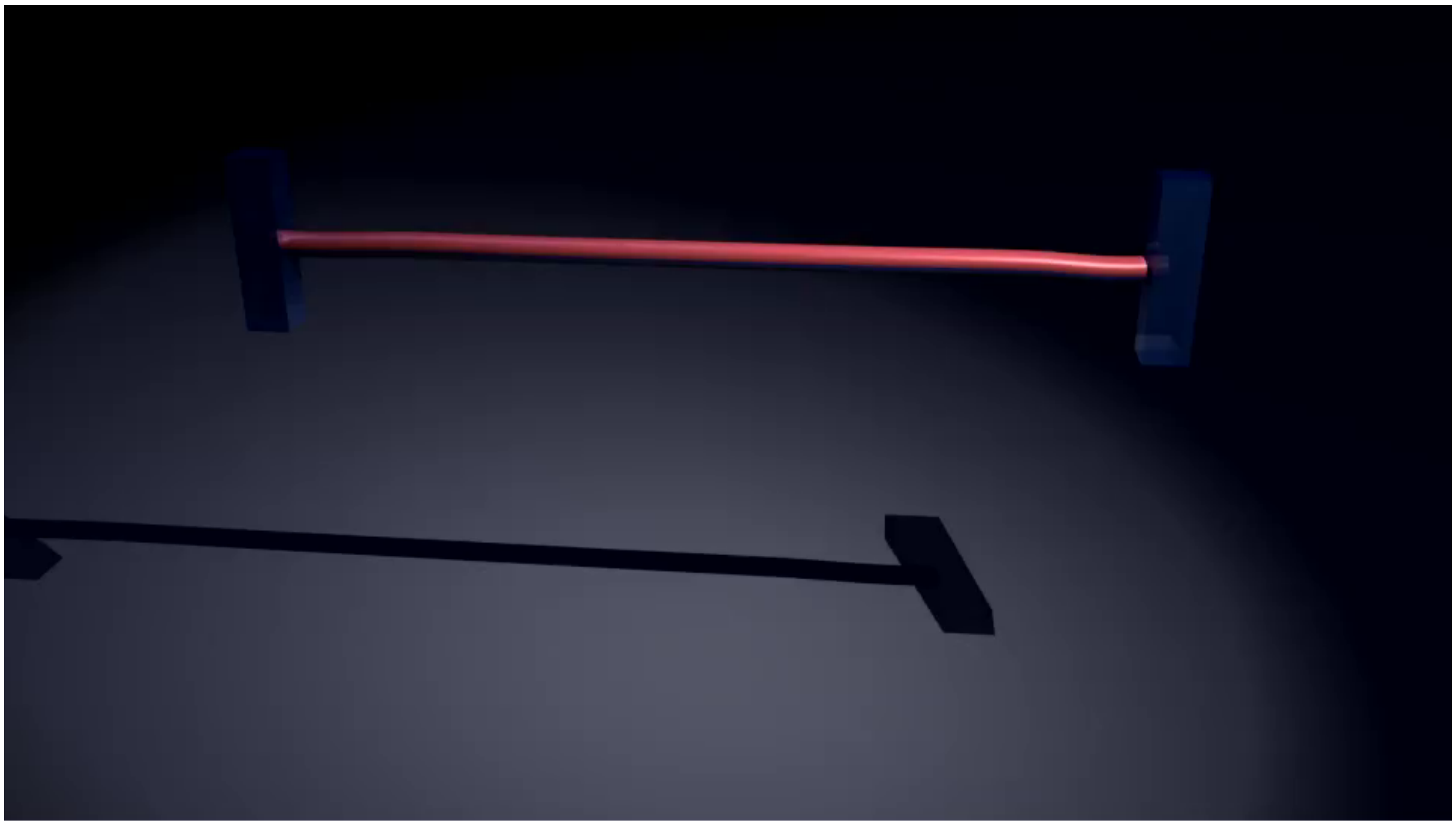
Shape Matching+Twisting Force [Rungiratananon et al. 2012]

- Too soft and unstable



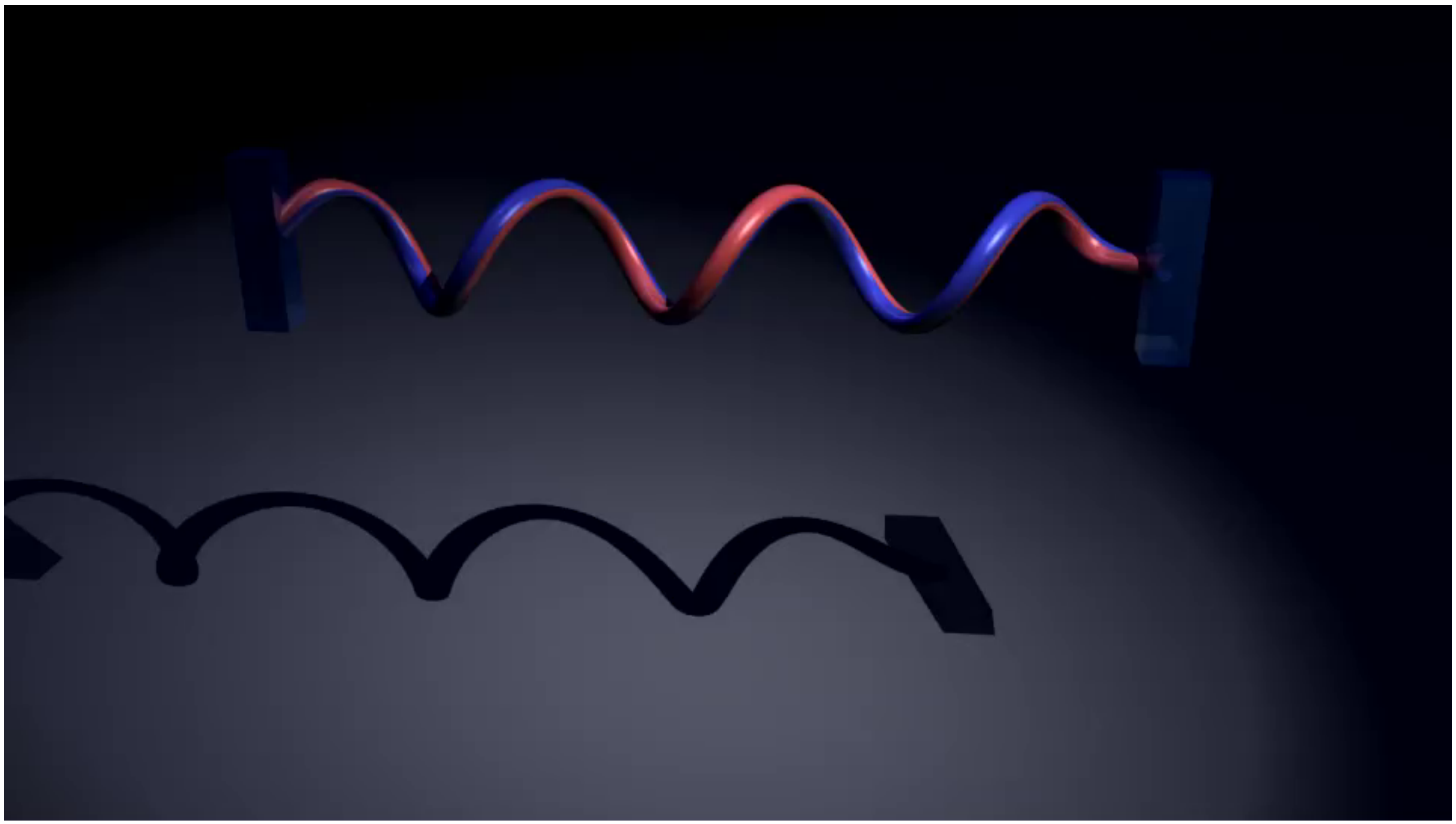
Oriented Particles [Müller et al. 2011]

- No coupling between twist and bending



Oriented Particles [Müller et al. 2011]

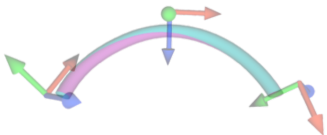
- No coupling between twist and bending



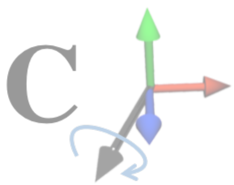
Comparison Summary

	Stability	Convergence	Twist +Bending	Time per Frame
Our technique	✓	✓	✓	1.1ms
Shape Matching +Twisting Force	✗	✗	✓	4.9ms
Oriented Particle	✓	✓	✗	5.0ms

Outline

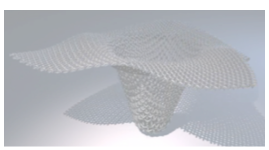


Twist representation for PDB



Constraints for twist and bending

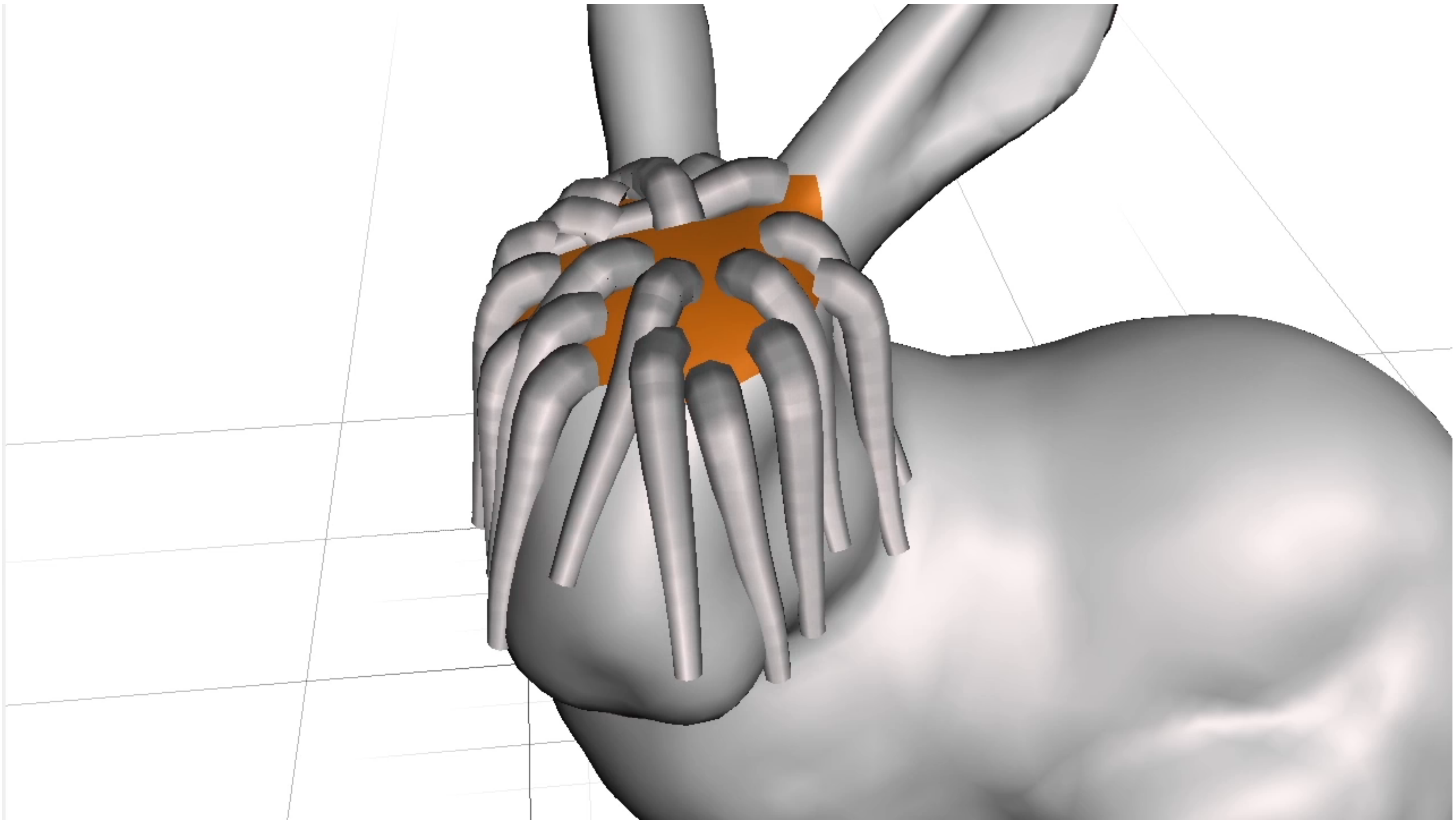
Handling vector type constraint



Result

Discussion

Application: Hair Design



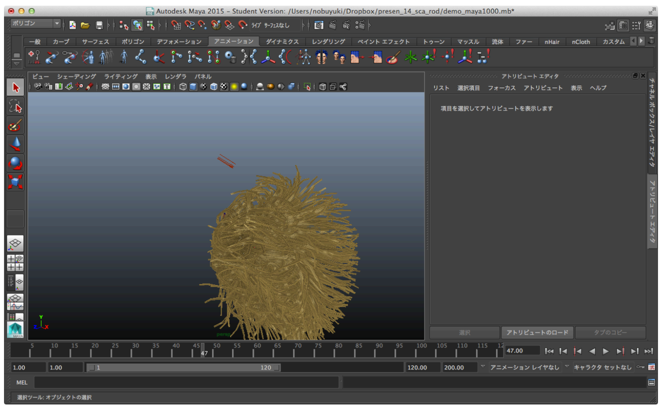
Future Work

- Preventing flipping completely
- Mathematical proof for stability
- Applications for vector type constraints
  - Continuum mechanics [Muller et al 2014]

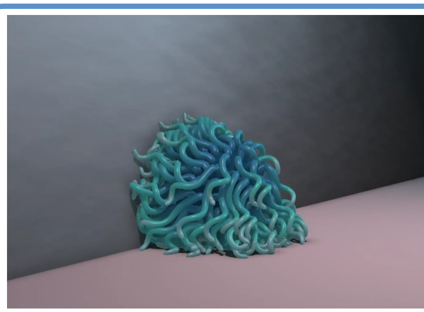
$$F^T F \rightarrow I$$

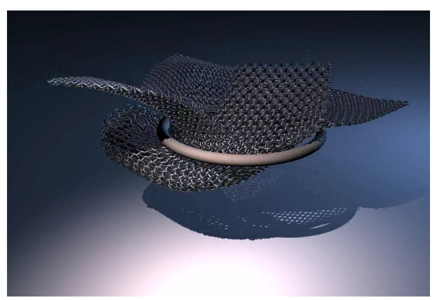
- **Summary**
  - First PBD approach for elastic rod simulation
- **Contribution**
  - ***Ghost points*** for defining frame using positions
  - New constraints formula for twist and bending
  - ***Variational Interpretation*** for vector constraint enforcement

- **Demo is available**
  - Autodesk MAYA 2015: Nucleus nHair
    - Bend with ”twist tracking” option



Live Demo!





1-3 FPS
(include rendering)

- **Paper is available**
  - Symposium on Computer Animation (SCA 2014)

Eurographics/ ACM SIGGRAPH Symposium on Computer Animation (2014)
Vladlen Koltun and Eftychios Sifakis (Editors)

Position-based Elastic Rods

Nobuyuki Umetani Ryan Schmidt Jos Stam

Autodesk Research

Abstract

We present a novel method to simulate complex bending and twisting of elastic rods. Elastic rods are commonly simulated using force based methods, such as the finite element method. These methods are accurate, but do not directly fit into the more efficient position-based dynamics framework, since the definition of material frames are not entirely based on positions. We introduce ghost points, which are additional points defined on edges, to naturally endow continuous material frames on discretized rods. We achieve robustness by a novel discretization of the Cosserat theory. The method supports coupling with a frame, a triangle, and a rigid body at the rod's end point. Our formulation is highly efficient, capable of simulating hundreds of strands in real-time.

Categories and Subject Descriptors (according to ACM CCS): I.6.8 [Computer Graphics]: Simulation and modeling—Animation

1. Introduction

Position-based dynamics (PBD) has been widely accepted in the field of computer animation due to its efficiency, robustness and simplicity. The goal of the PBD is not to simulate physics as accurately as possible, but rather to sacrifice some quantitative accuracy to generate visually plausible simulation results very quickly. To this end, PBD has broadly been applied in many game engines and visual effects, where speed and controllability is crucial. PBD has primarily been used to simulate various physical phenomena associated with solid and thin-shell (i.e., clothing) deformations [MHHR07]. We present a new application of PBD to

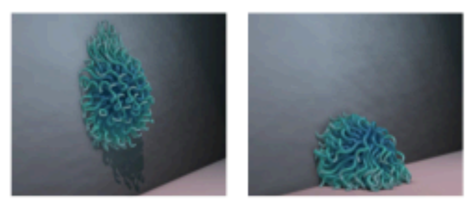
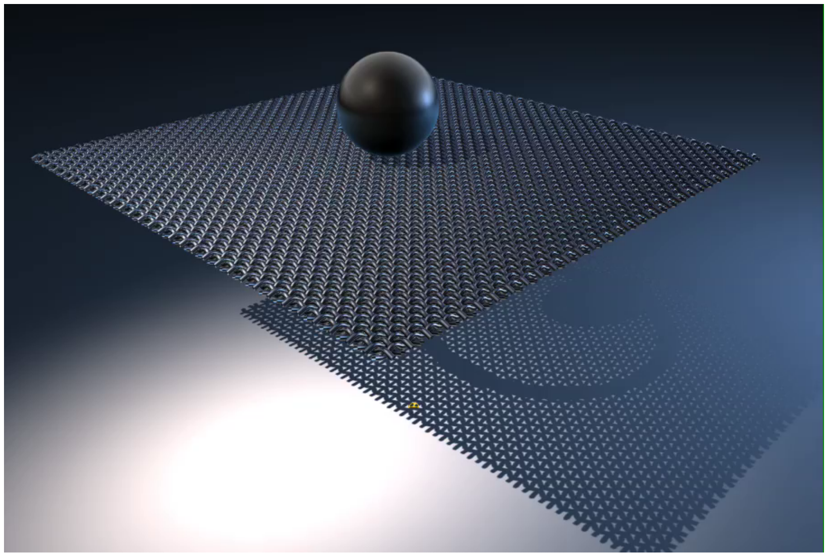
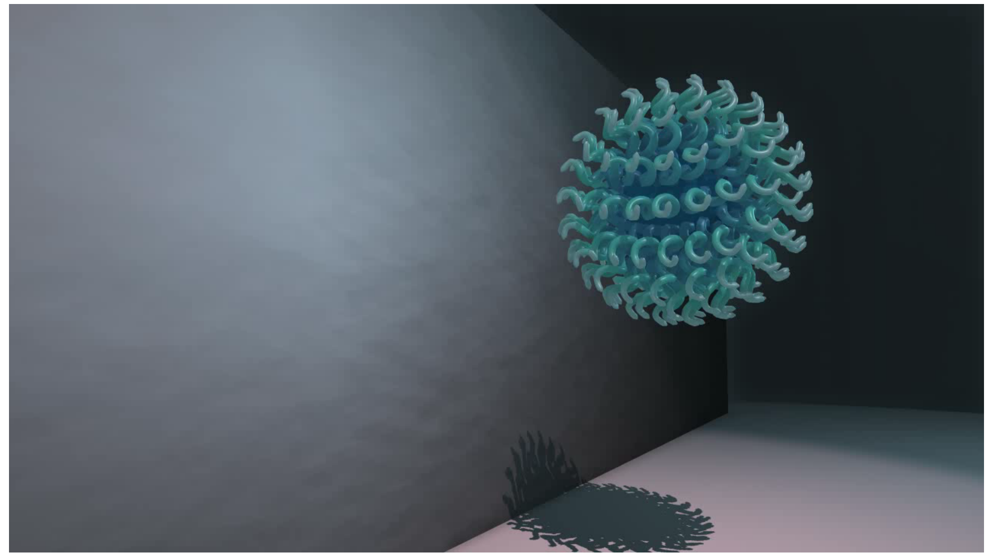


Figure 1: *A squishy ball hits a wall. Tentacles of the squishy ball was modeled with our position-based elastic rods.*

Representation of twist in elastic rods is actively researched, but so far no previous work has successfully

Thanks!   Any Questions?

- **Acknowledgements**
  - Duncan Brinsmead
  - Anonymous reviewers

Variational Interpretation of Constraint Enforcements

$$\Delta p = \arg\min_{\Delta p} \sum_{s \in S} m_s \left| \Delta p_s \right|^2, \quad where \quad C(p + \Delta p) = 0$$

Lagrange multiplier

$$\nabla_s \left(\sum_{t \in S} m_t \left| \Delta p_t \right|^2 + \lambda^T C \right) = 0 \quad (\forall s \in S)$$

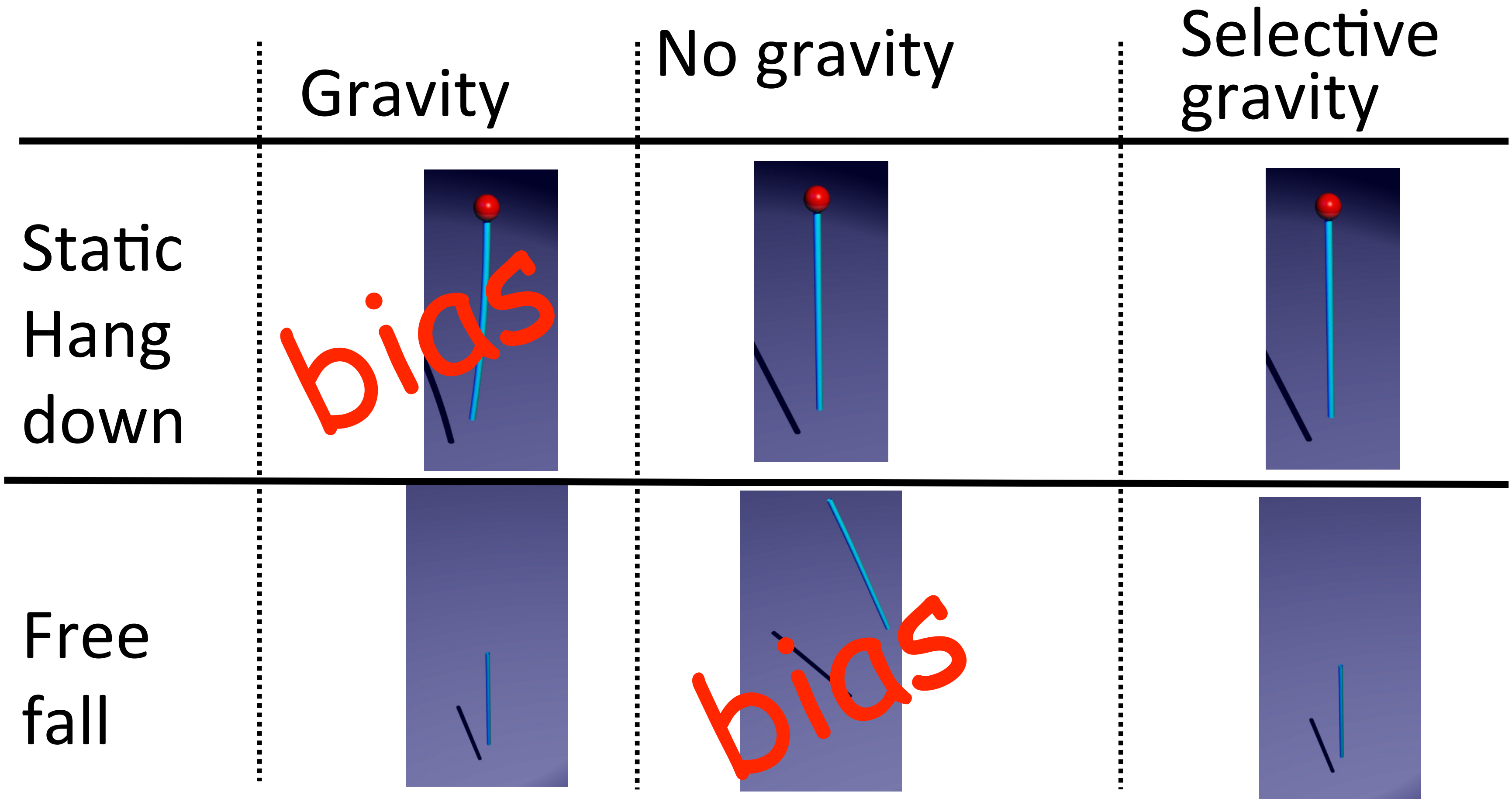
$$C + \nabla C \Delta p = 0$$

$$\Delta p_s = -\frac{1}{m_s} \nabla_s C^T \frac{\lambda}{2}$$

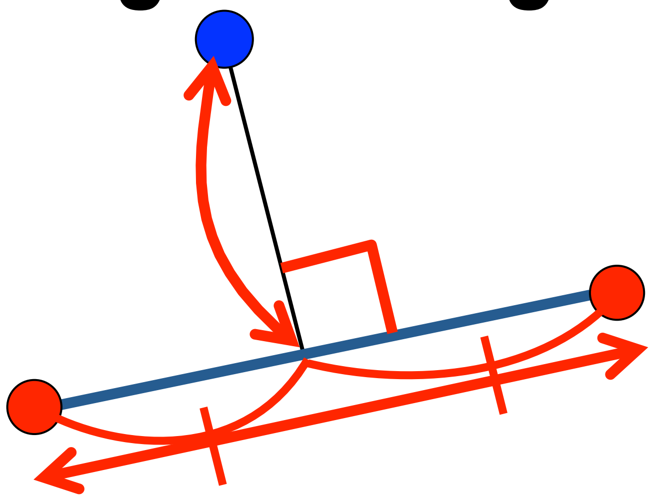
$$\lambda = 2 \left(\sum_{t \in S} \frac{1}{m_t} \nabla_t C^T \nabla_t C \right)^{-1} C$$

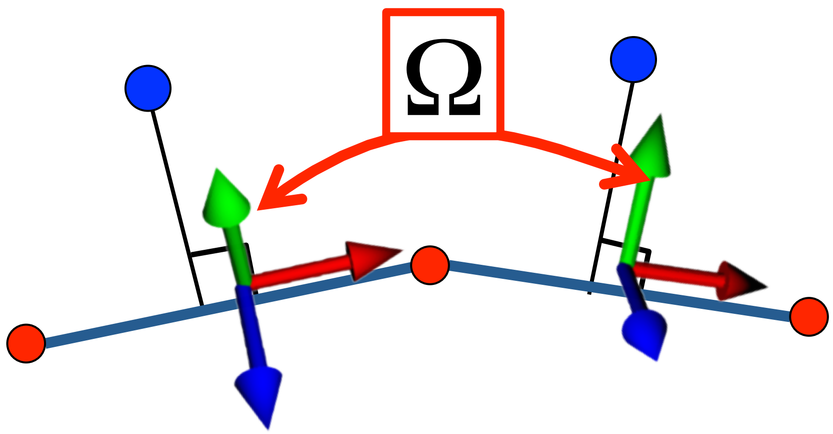
$$\Delta p_s = -\frac{1}{m_s} \nabla_s C^T \left(\sum_{t \in S} \frac{1}{m_t} \nabla_t C^T \nabla_t C \right)^{-1} C$$

Removing Bias of Gravity on Ghost Points



Putting it All Together





- Edge length constraint
- Perpendicular bisector constraint
- Distance constraint

Vector-type constraints

- Bending & twisting constraint