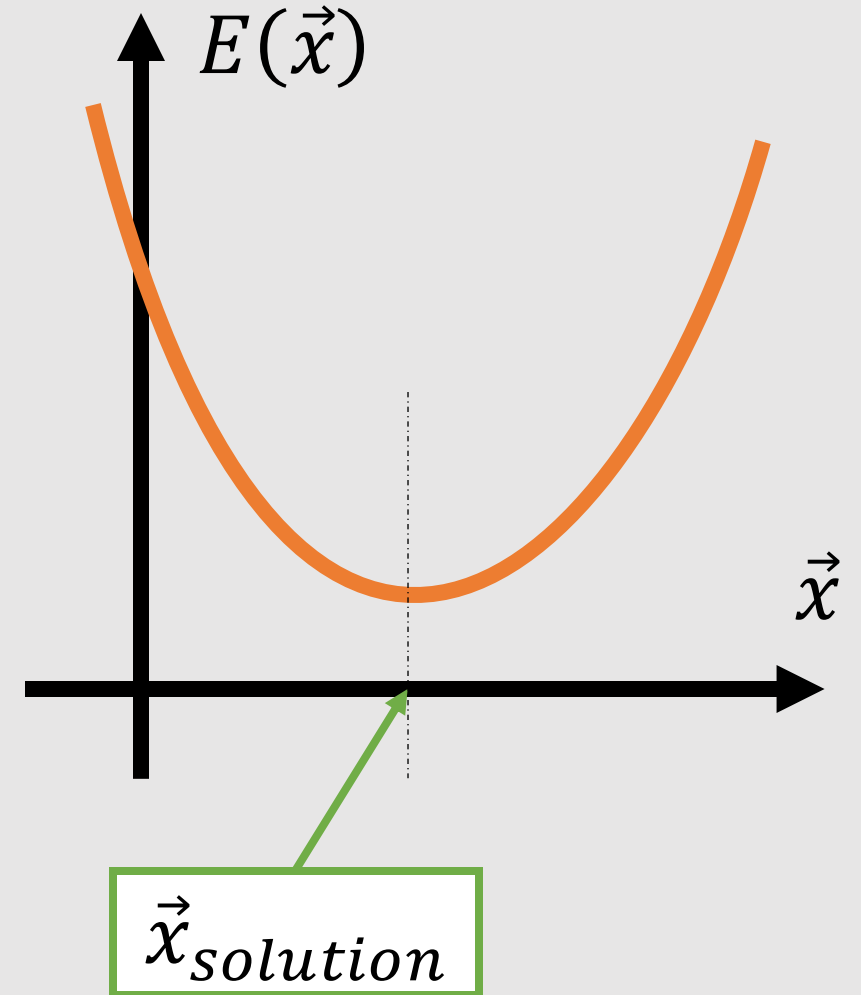


Variational Backward Euler Time Integration

What is **Variational** Method?

- Solution is expressed by the optimization

$$\vec{X}_{solution} = \operatorname{argmin}_{\vec{x}} E(\vec{x})$$



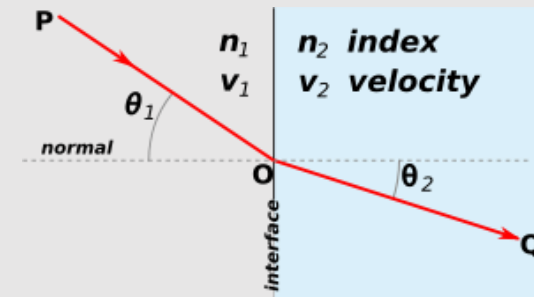
Variational Principles in Physics

- Mechanics



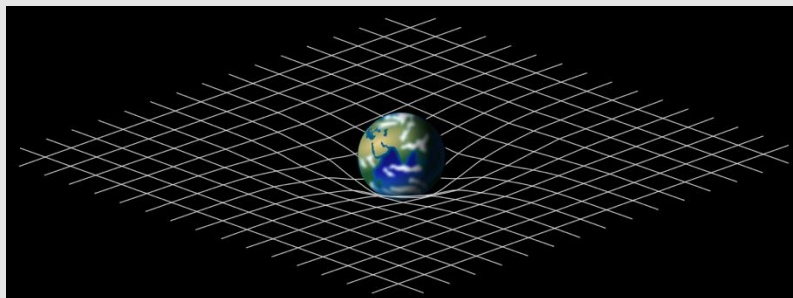
(Wikipedia)

- Optics



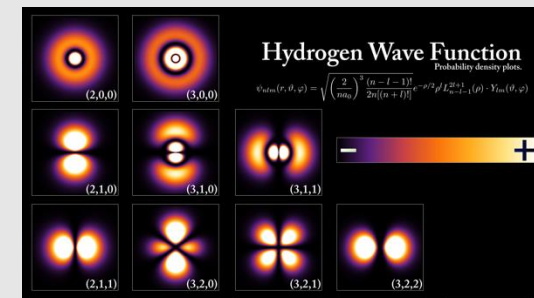
(Wikipedia)

- General relativity



(Wikipedia)

- Quantum physics

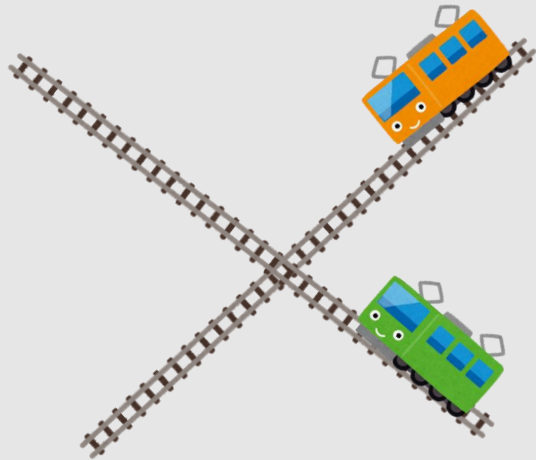


(Wikipedia)

Solving Constraints v.s. Variational Problem



Solution should be
on this line



Linearization

$$Ax = b$$

Solution should be at the
bottom of this hole



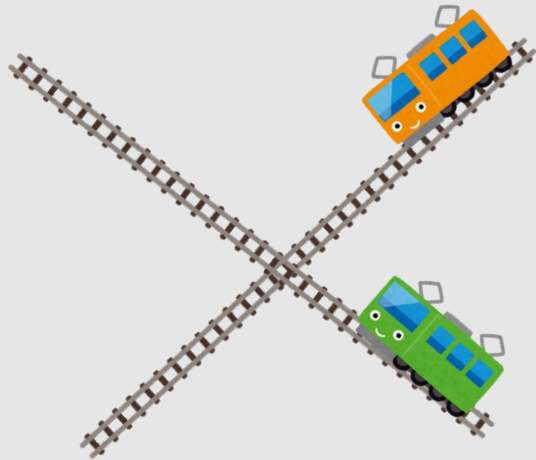
There are many
weapons to fight



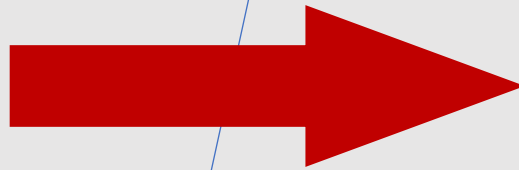
Solving Constraints v.s. Variational Problem



Solution should be
on this line



Solution should be at the
bottom of this hole



How can we convert the problem?

Making a Variational Problem

- We only need a single scalar value E to find solution

$A\vec{x} = \vec{b}$

L2 norm of residual

$$E(\vec{x}) = \|\vec{A}\vec{x} - \vec{b}\|^2$$

integration

$$E(\vec{x}) = \frac{1}{2} \vec{x}^T A \vec{x} - \vec{b}^T \vec{x}$$

Making a Variational Problem

- Integration with \vec{x} will make a variational formula


$$\frac{\partial W(\vec{x})}{\partial \vec{x}} = \vec{b} \quad \xrightarrow{\text{integration}} \quad E(\vec{x}) = W(\vec{x}) - \vec{b}^T \vec{x}$$

$$\frac{\partial W(\vec{x})}{\partial \vec{x}} = -M\vec{x} \quad \xrightarrow{\text{integration}} \quad E(\vec{x}) = W(\vec{x}) + \frac{1}{2} \vec{x}^T M \vec{x}$$


Variational Formulation of Backward Euler

- Review of Backward Euler

$$\frac{ds}{dt} = \frac{s_{i+1} - s_i}{dt} = F(s_{i+1})$$



plug in $s_i = \begin{pmatrix} \vec{v}_i \\ \vec{x}_i \end{pmatrix}$, $M\dot{\vec{v}} = \frac{\partial W}{\partial \vec{x}}$



$$\begin{cases} \vec{x}_{i+1} = \vec{x}_i + dt \cdot \vec{v}_i + dt^2 \cdot M^{-1} \frac{\partial W}{\partial x_{i+1}} \\ \vec{v}_{i+1} = (\vec{x}_{i+1} - \vec{x}_i)/dt \end{cases}$$

Variational Formulation of Backward Euler

- Getting next time step by minimization

integration

$$\vec{x}_{i+1} = \operatorname{argmin}_{\vec{x}} E_i(\vec{x})$$

$$E_i(\vec{x}) = W(\vec{x}) + \frac{1}{2dt^2} (\vec{x} - \vec{x}_i)^T M (\vec{x} - \vec{x}_i)$$

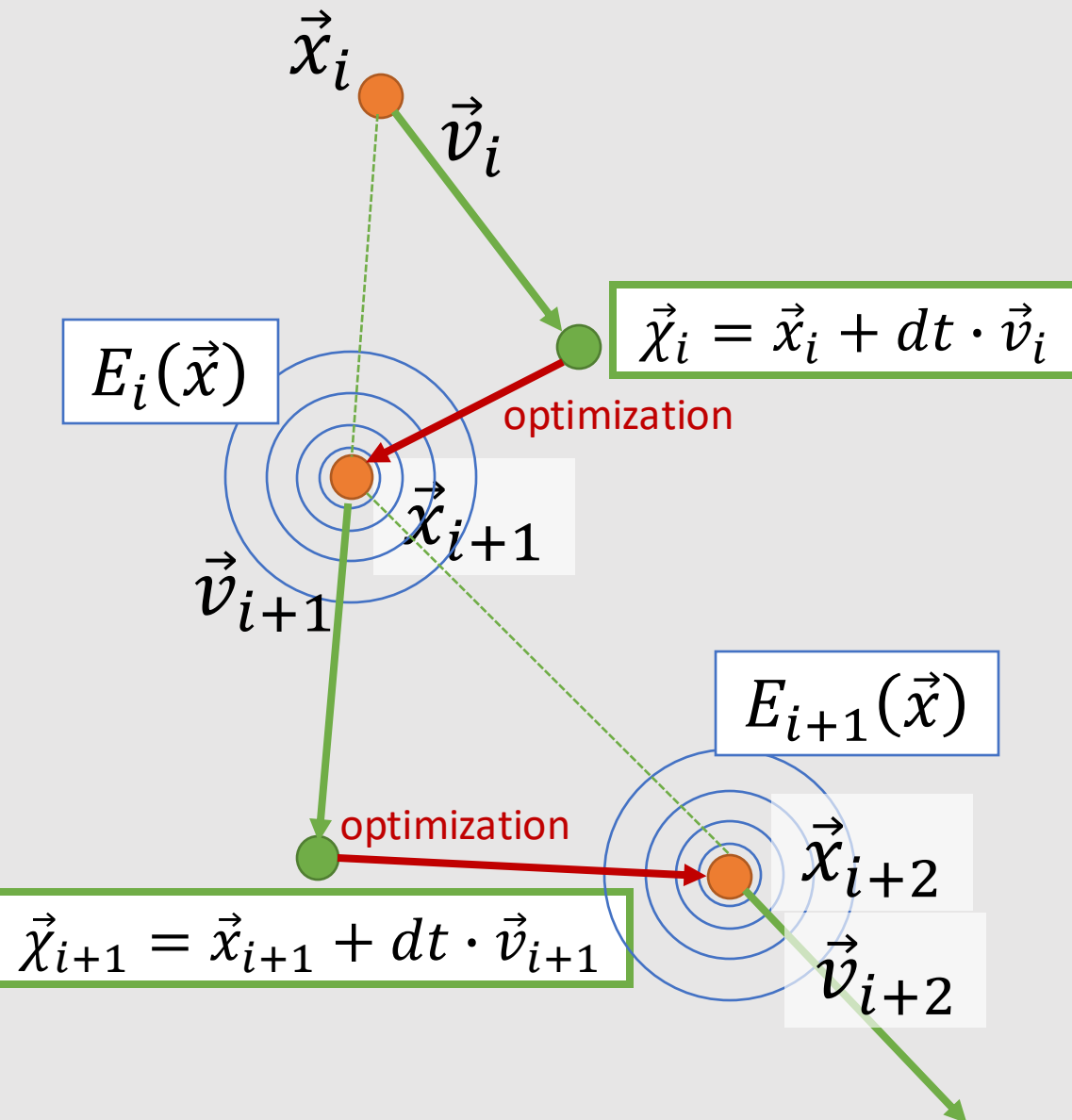
$$\vec{x}_i = \vec{x}_i + dt \cdot \vec{v}_i$$

$$\vec{x}_{i+1} = \vec{x}_i + dt \cdot \vec{v}_i + dt^2 \cdot M^{-1} \frac{\partial W}{\partial \vec{x}_{i+1}}$$

$$\vec{v}_{i+1} = (\vec{x}_{i+1} - \vec{x}_i) / dt$$



Scheme of Variational Backward Euler



1. compute temporary position

$$\vec{\tilde{x}}_i = \vec{x}_i + dt \cdot \vec{v}_i$$

2. optimize $E_i(\vec{x})$ to get \vec{x}_{i+1}

3. Set velocity

$$\vec{v}_{i+1} = \frac{(\vec{x}_{i+1} - \vec{x}_i)}{dt}$$

4. Goto 1



Variational Formula Explained

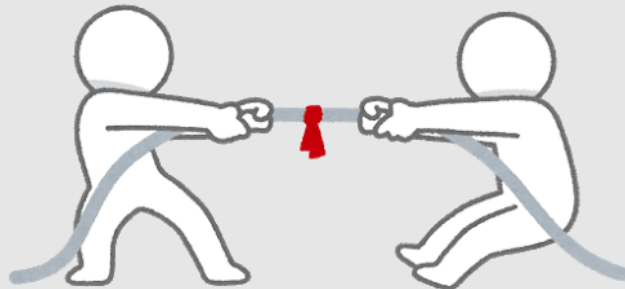
- Solving tradeoff between elasticity & inertia

$$E_i(\vec{x}) = \underbrace{W(\vec{x})}_{\text{elasticity}} + \underbrace{\frac{1}{2dt^2} (\vec{x} - \vec{\chi}_i)^T M (\vec{x} - \vec{\chi}_i)}_{\text{inertia}}$$

$\vec{\chi}_i = \vec{x}_i + dt \cdot \vec{v}_i$

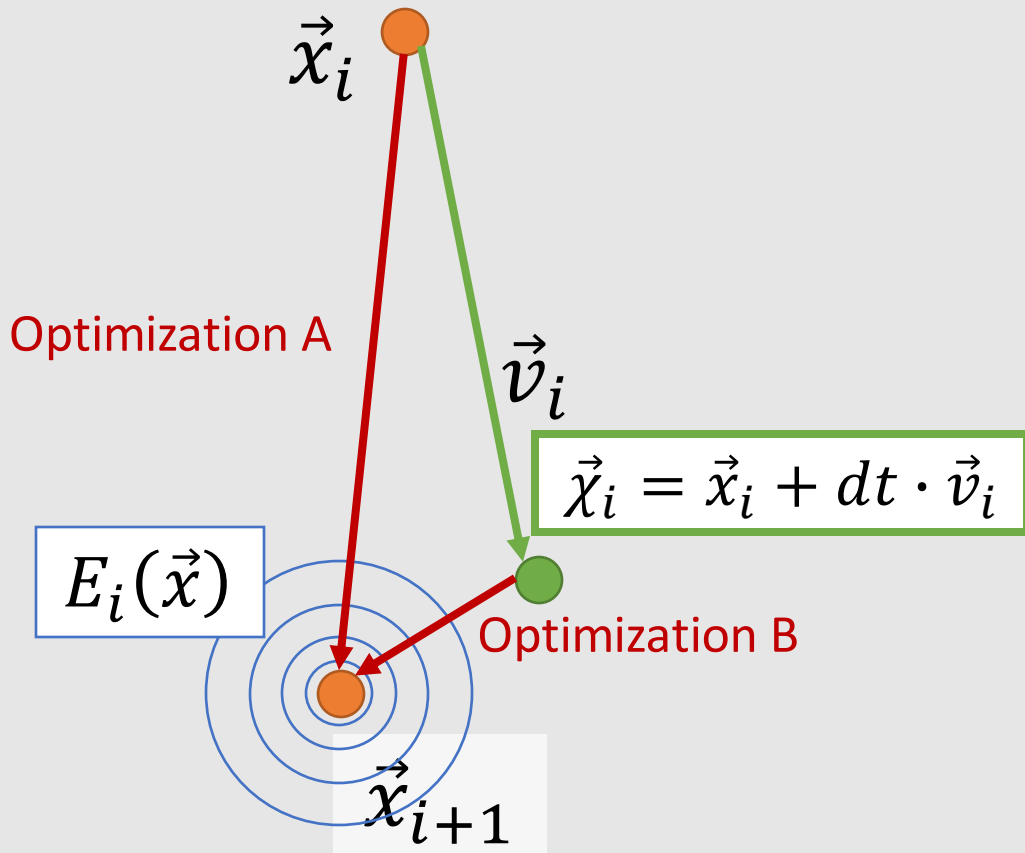
Trying to “undeform” shape

Trying to move shape with velocity \vec{v}_i



Optimization with Newton Method

- optimize $E_i(\vec{x}) = W(\vec{x}) + 1/2 dt^2 (\vec{x} - \vec{\chi}_i)^T M(\vec{x} - \vec{\chi}_i)$ to get \vec{x}_{i+1}



Optimization A (bad ☹️)

$$\vec{x}_{i+1} = \vec{x}_i - \left[\frac{\partial^2 W(\vec{x}_i)}{\partial^2 \vec{x}} \right]^{-1} \left(\frac{\partial W(\vec{x}_i)}{\partial \vec{x}} \right)$$

Optimization B (good 😊)

$$\vec{x}_{i+1} = \vec{\chi}_i - \left[\frac{\partial^2 W(\vec{\chi}_i)}{\partial^2 \vec{x}} \right]^{-1} \left(\frac{\partial W(\vec{\chi}_i)}{\partial \vec{x}} \right)$$