Einstein's Summation Rule

- Repeated indices are summed over

I have made a great discovery in mathematics!

inner product

$$a \cdot b = a_i b_i = \sum_i a_i b_i$$

Frobenius inner product

$$\langle A, B \rangle_F = A_{ij} B_{ij} = \sum_j \sum_i A_{ij} B_{ij}$$

Let's Practice Einstein's Summation Rule

$$tr(A) = ?$$

$$(A^T B)_{ij} = ?$$

$$tr(A^T B) = ?$$

$$\left(\vec{a}^T \vec{b} I\right)_{ij} = ?$$

$$a_i b_i = ?$$

$$a_{ii} a_{jj} = ?$$

$$a_{ii} a_{ii} = ?$$

Frobenius Inner Product $\langle A, B \rangle_F = A_{ij} B_{ij}$

$$A = [\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n]$$
$$B = [\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_n]$$

$$\langle A, B \rangle_F = \sum_i \vec{a}_i^T \vec{b}_i$$

$$\langle RA, B \rangle_F = R_{ik} A_{kj} B_{ij} = \sum_i (R\vec{a}_i)^T \vec{b}_i$$

$$= A_{kj}(R^T B)_{kj} = \langle A, R^T B \rangle_F = \sum_i \vec{a}_i^T R \vec{b}_i$$

$$= R_{ik}(BA^T)_{ik} = \langle R, BA^T \rangle_F = \left\langle R, \sum_i \vec{b}_i \otimes \vec{a}_i \right\rangle_F$$
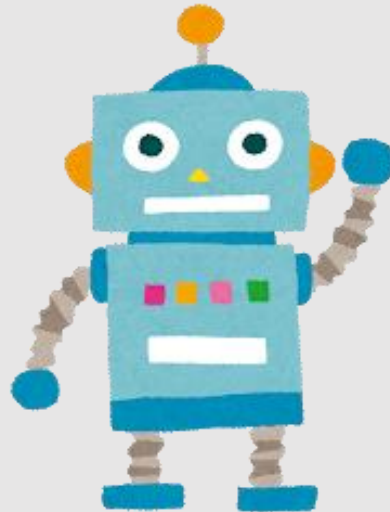
Tensor

What is Tensor?

In mathematics, a tensor is an algebraic object that **describes a (multilinear) relationship** between sets of algebraic objects related to a vector space.

https://en.wikipedia.org/wiki/Tensor

scalar a
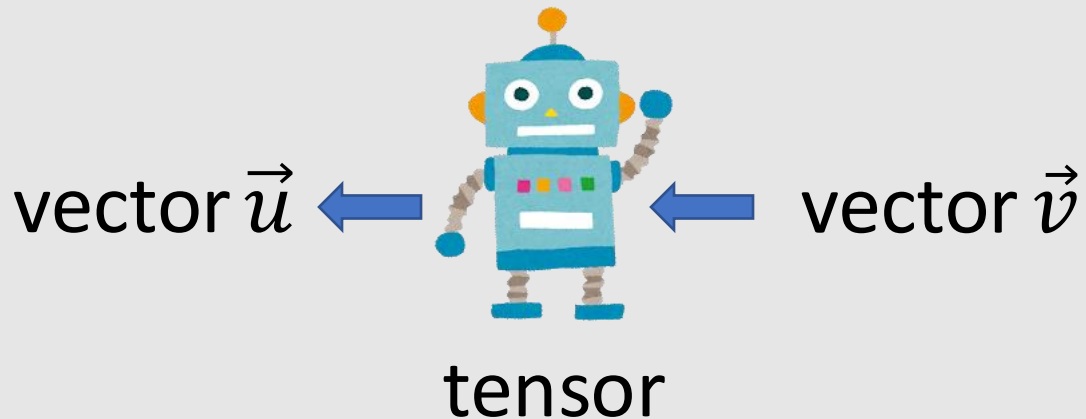
vector $\vec{u}$

tensor A

scalar b

vector $\vec{v}$

tensor B

tensor

Two ways to Understand 2nd-order Tensor

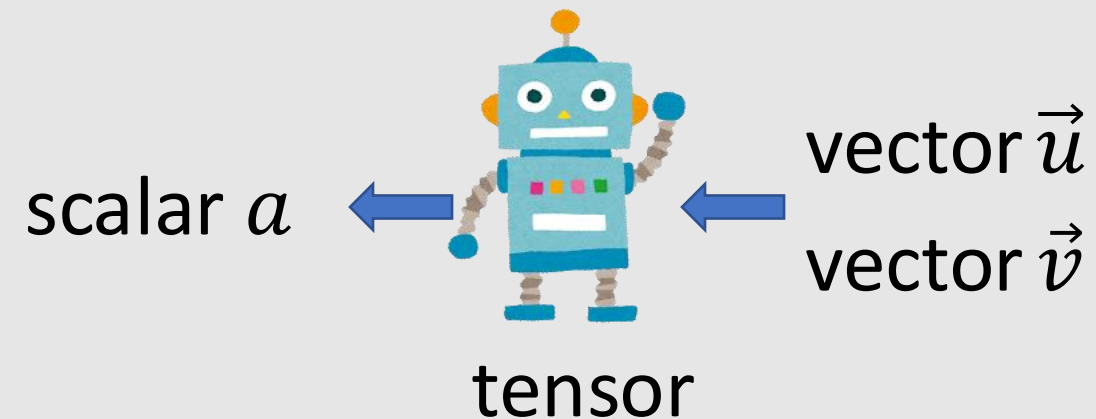- Transformation by a tensor is give by the inner product

Linear form

$$\vec{u} = A \cdot \vec{v}$$

Quadratic form

$$a = \vec{u} \cdot (A \cdot \vec{v})$$

vector $\vec{u}$ ← [robot] ← vector $\vec{v}$

tensor

scalar a ← [robot] ← vector $\vec{u}$ vector $\vec{v}$

tensor

Outer Product (Tensor Product)

- Outer product makes a tensor from two vectors

$$\vec{a} \otimes \vec{b} \implies \left(\vec{a} \otimes \vec{b} \right) \cdot \vec{u} = \vec{a}(\vec{b} \cdot \vec{u})$$

- Tensor product $\vec{e} \otimes \vec{e}$ $(\|e\| = 1)$ defines projection

check it out!

Definition

Projection $\mathcal{P}$

$$\mathcal{P}\big(\mathcal{P}(x)\big) = \mathcal{P}$$

Outer Product (Tensor Product)

- Transformation for vectors in the outer product

check it out!

$$(A\vec{a})\otimes(B\vec{b})=?$$

Definition

$$\vec{a}\otimes\vec{b}$$

$$\left(\vec{a}\otimes\vec{b}\right)\cdot\vec{u}=\vec{a}(\vec{b}\cdot\vec{u})$$

Tensor + Basis = Matrix

- Inner product with a basis vector gives a coefficient
  - This is true even if the basis is not orthonormal

$$v_i = \vec{v} \cdot \vec{e}_i$$

$$a_{ij} = \vec{e}_i \cdot (A \cdot \vec{e}_j)$$

Tensor & Matrix: Common Misunderstanding

Orthonormal Coordinates

- Tensor can be written with bases and coefficients

$$e_i \cdot e_j = \delta_{ij}$$

$$v_i = \vec{v} \cdot \vec{e}_i$$

$$\Longrightarrow \vec{v} = v_i \vec{e}_i$$

$$a_{ij} = \vec{e}_i \cdot (A \cdot \vec{e}_j)$$
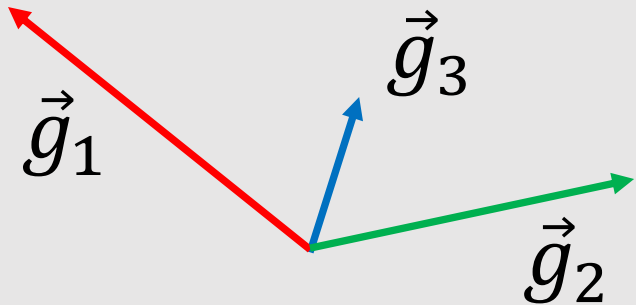
$$\Longrightarrow A = a_{ij} \vec{e}_i \otimes \vec{e}_j$$
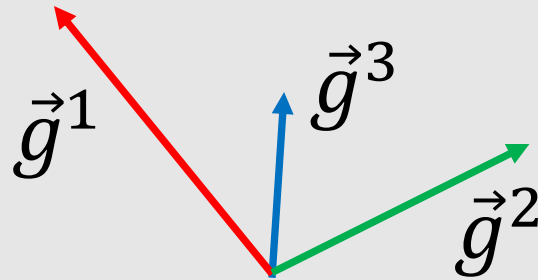
check it out!

Curvilinear Coordinates

- Non-orthogonal and un-normalized bases
- Dual bases solve the problem

$$\vec{g}_i \cdot \vec{g}_j \neq \delta_{ij}$$

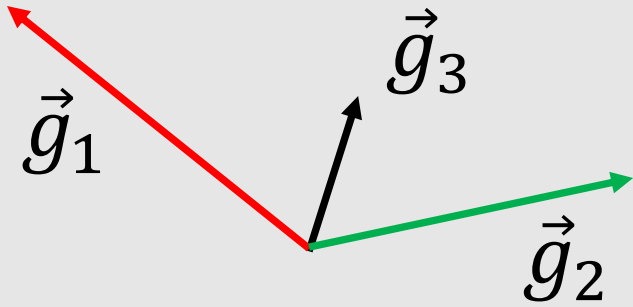dual basis

$$\vec{g}_i \cdot \vec{g}^j = \delta_i^j$$

$$\vec{g}^1 = \frac{\vec{g}_2 \times \vec{g}_3}{\vec{g}_2 \cdot (\vec{g}_2 \times \vec{g}_3)}$$

$$\vec{g}^2 = \frac{\vec{g}_3 \times \vec{g}_1}{\vec{g}_3 \cdot (\vec{g}_3 \times \vec{g}_1)}$$

$$\vec{g}^3 = \frac{\vec{g}_1 \times \vec{g}_2}{\vec{g}_1 \cdot (\vec{g}_1 \times \vec{g}_2)}$$

Curvilinear Coordinates

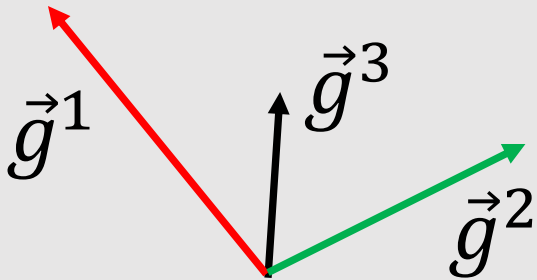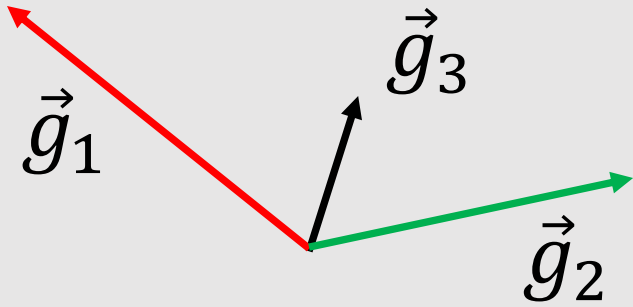- Expression of a vector in curvilinear coordinates

$$v^i = \vec{v} \cdot \vec{g}^i \qquad \Longrightarrow \qquad \vec{v} = v^i \vec{g}_i$$

dual basis $\quad g_i \cdot g^j = \delta_i^j$

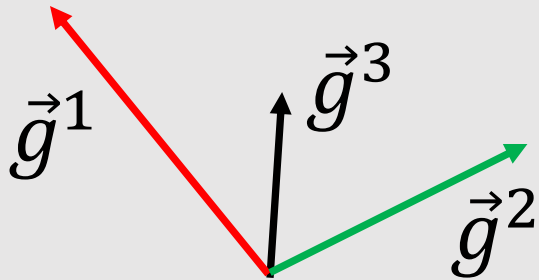$$v_i = \vec{v} \cdot \vec{g}_i \qquad \Longrightarrow \qquad \vec{v} = v_i \vec{g}^i$$

Curvilinear Coordinates

- Expression of a tensor in curvilinear coordinates

dual basis

$$g_i \cdot g^j = \delta_i^j$$

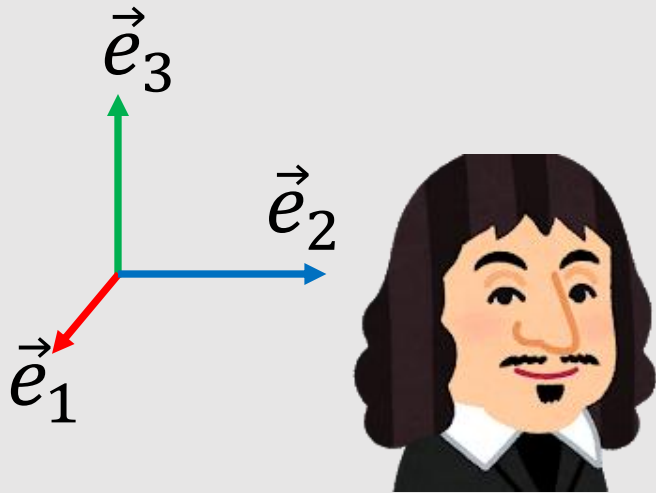$$a^{ij} = \vec{g}^i \cdot (A \cdot \vec{g}^j) \implies A = a^{ij} \vec{g}_i \otimes \vec{g}_j$$

$$a_{ij} = \vec{g}_i \cdot (A \cdot \vec{g}_j) \implies A = a_{ij} \vec{g}^i \otimes \vec{g}^j$$

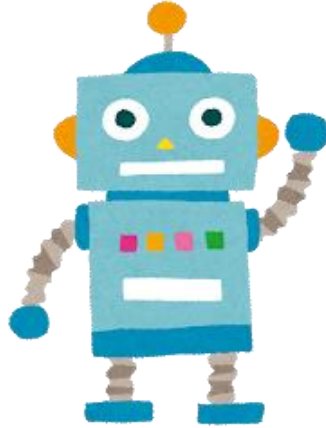$$a^i{}_j = \vec{g}^i \cdot (A \cdot \vec{g}_i) \implies A = a^i{}_j \vec{g}_i \otimes \vec{g}^j$$

$$a_i{}^j = \vec{g}_i \cdot (A \cdot \vec{g}^j) \implies A = a_i{}^j \vec{g}^i \otimes \vec{g}_j$$
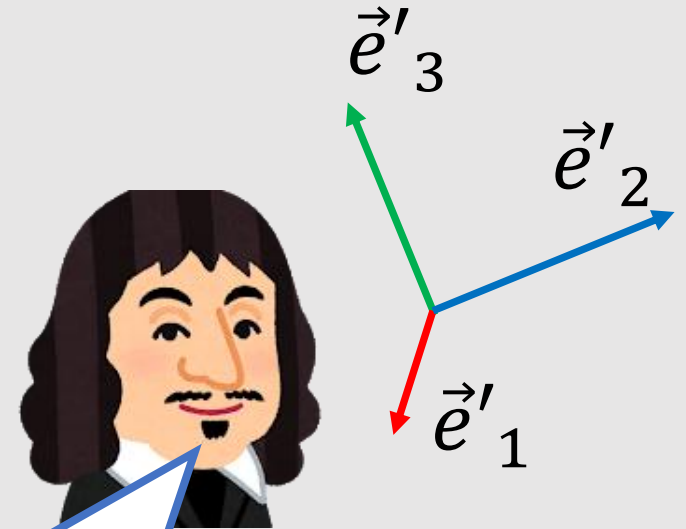
Coordinate Transformation

tensor: A

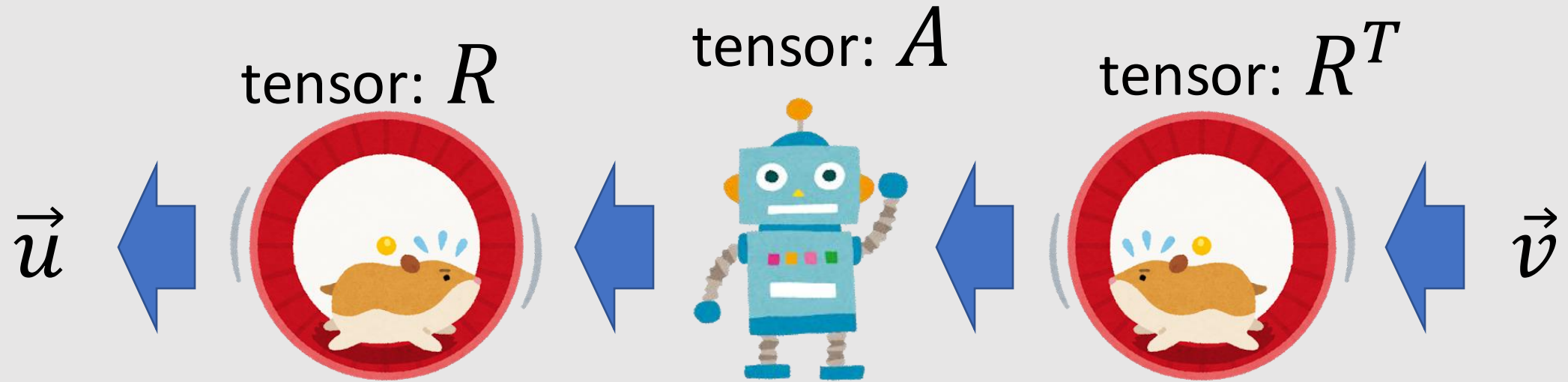$$a_{ij} = \vec{e}_i \cdot (A \cdot \vec{e}_j)$$

$$a'_{ij} = \vec{e}'_i \cdot (A \cdot \vec{e}'_j)$$

same tensor, different (coefficient) matrix!

Rotation of a Tensor

- Un-rotating input and rotating output

tensor: R tensor: A tensor: R^T

$$\vec{u} = RAR^T\vec{v}$$

Rotation of a Tensor

- Rotating bases while using the same coefficients

$$A = a_{ij}\vec{e}_i \otimes \vec{e}_j$$

rotation of basis

$$A' = a_{ij}(R\vec{e}_i) \otimes (R\vec{e}_j)$$
$$= a_{ij}R(\vec{e}_i \otimes \vec{e}_j)R^T$$
$$= R(a_{ij}\vec{e}_i \otimes \vec{e}_j)R^T$$
$$= RAR^T$$

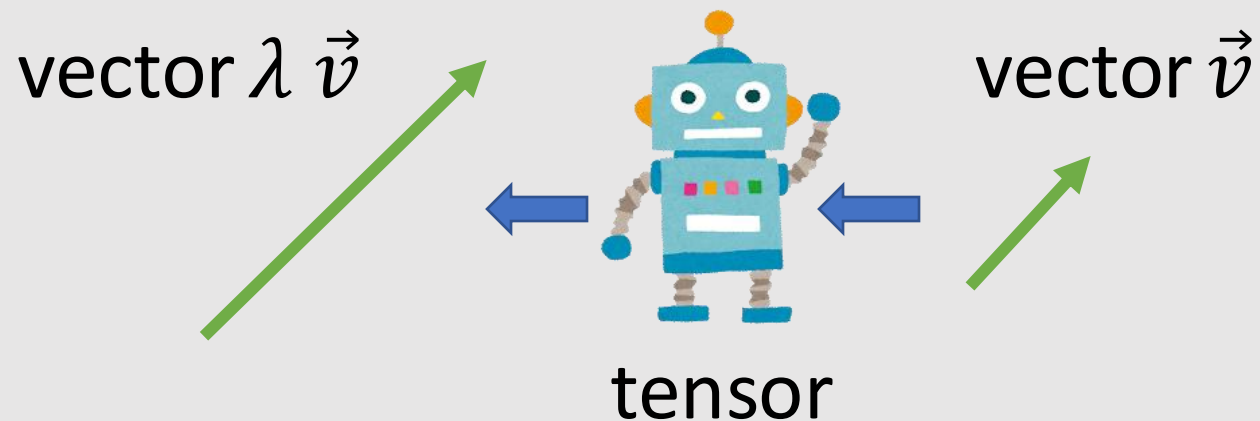check it out!

Simple Elastic Potential Energy for Continuum

Eigenvalue of Symmetric Tensor

- Eigenvalue of tensor is defined without matrix & coordinate

Linear form

$$\lambda \vec{v} = A \cdot \vec{v}$$

vector $\lambda \, \vec{v}$

vector $\vec{v}$

tensor

Eigenvalues and Frobenius Norm

from characteristic equation

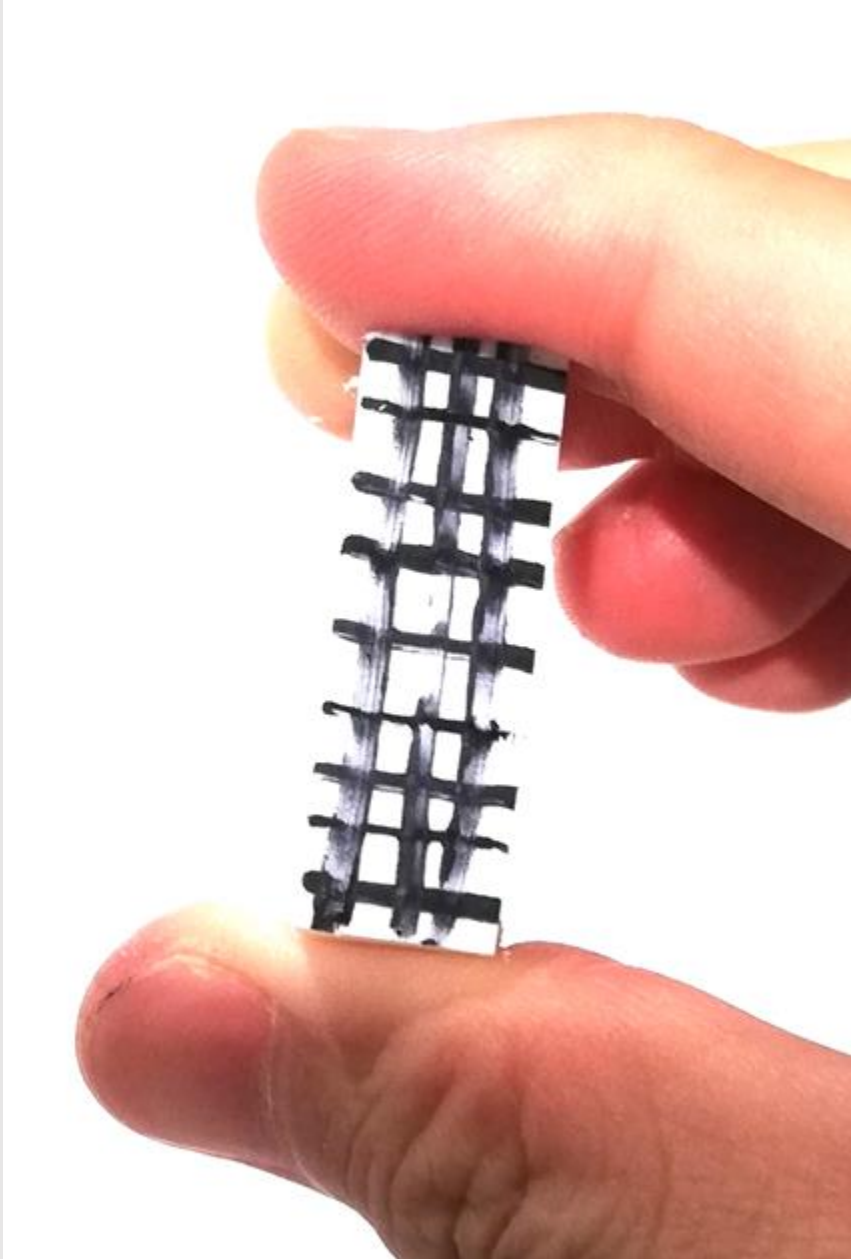$$\text{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$$

$$A^2 \vec{e} = \lambda^2 \vec{e}$$

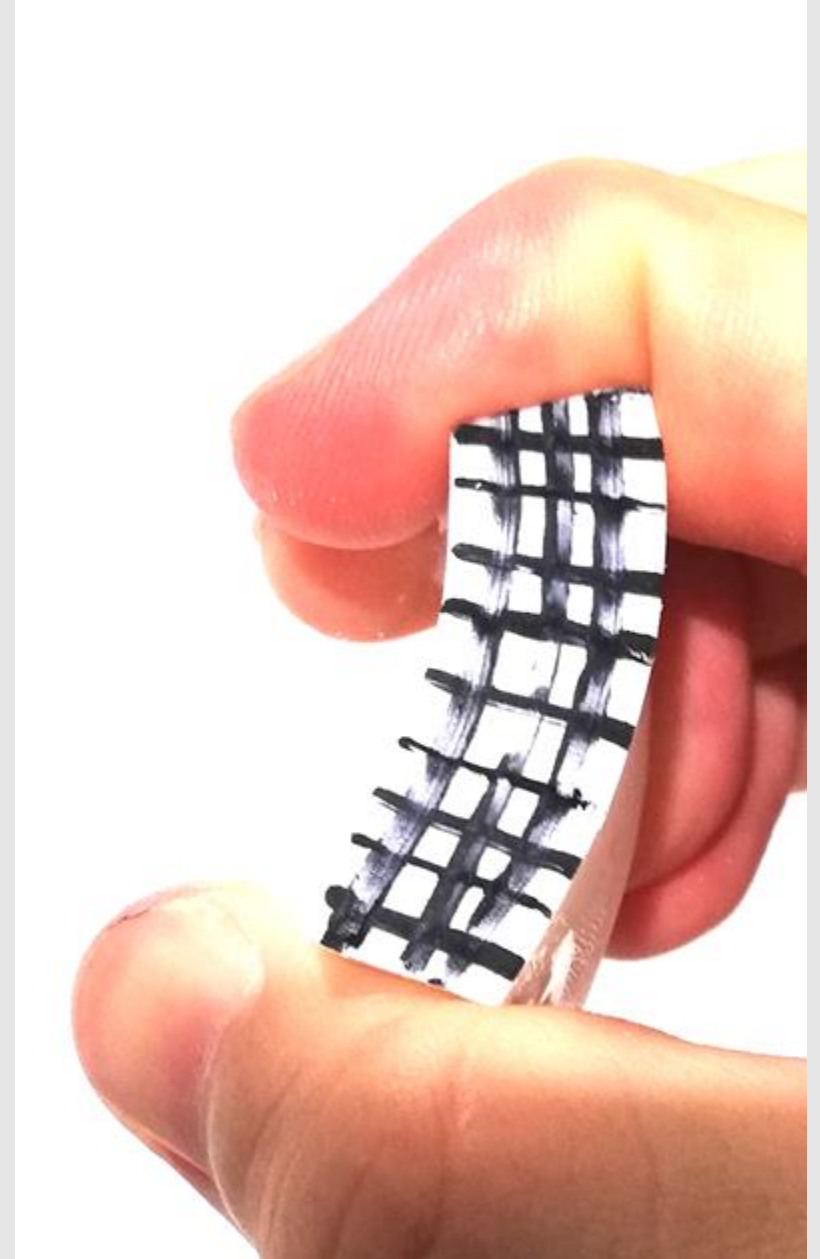$$\text{tr}(A^2) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

Frobenius norm

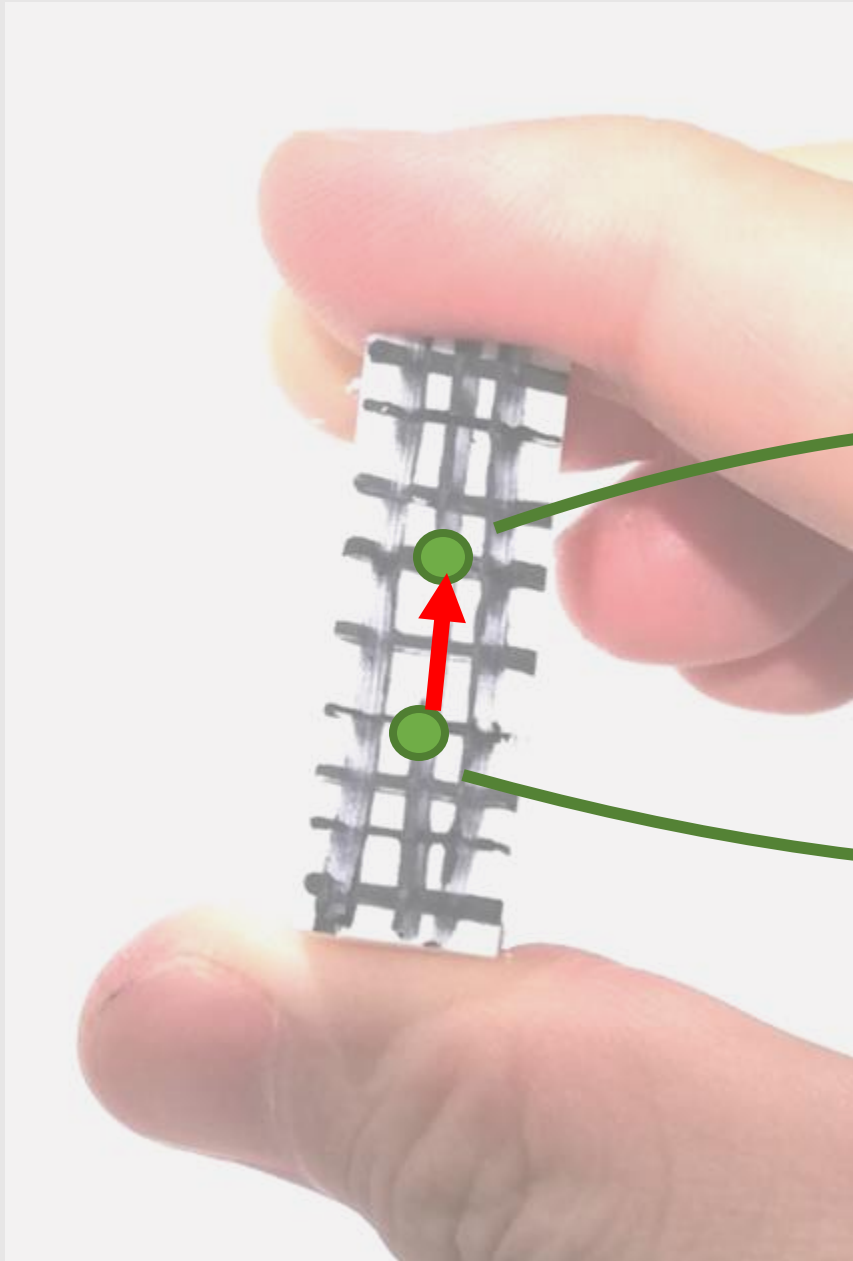$$= \text{tr}(A^T A) = \sum_{1 \le i,j \le 3} a_{ij}^2 = \|A\|_F^2$$
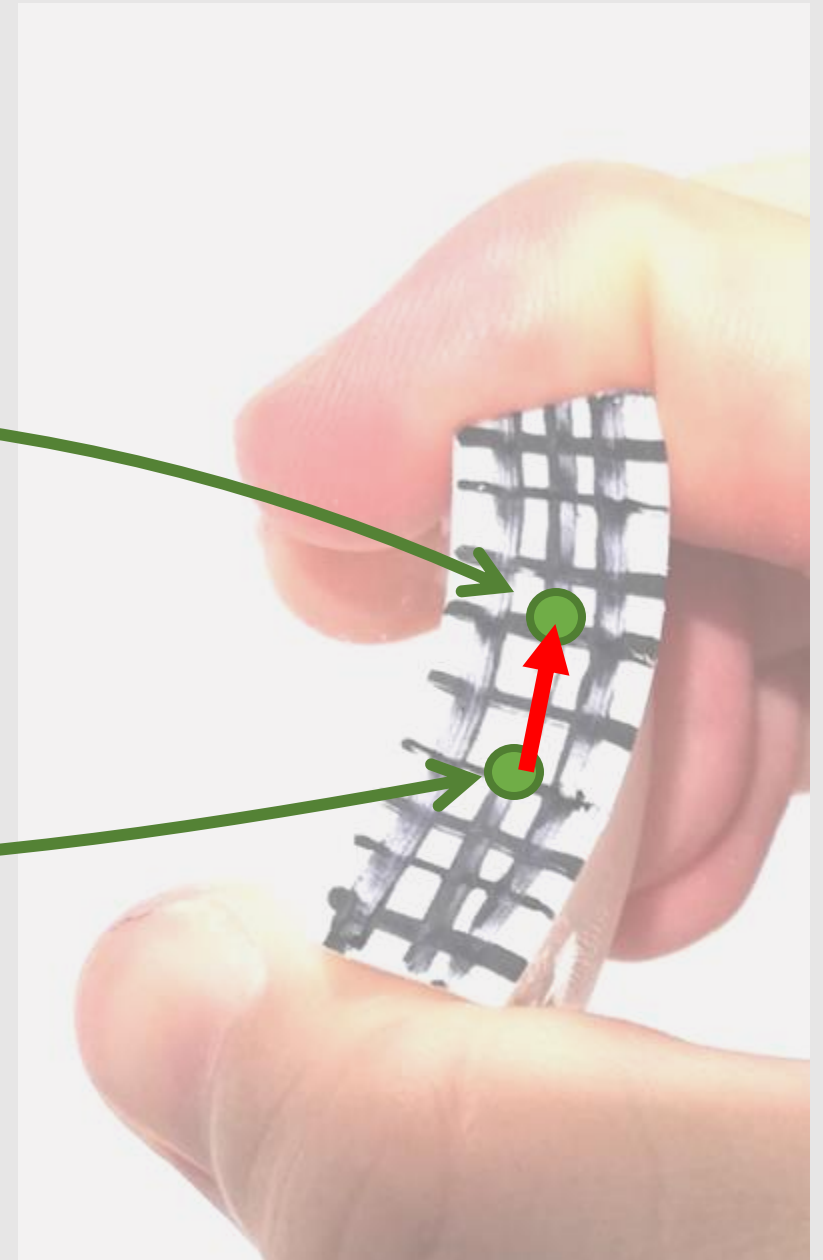
rest shape

deformed shape

rest shape

deformed shape

rest shape

deformed shape

F : deformation gradient tensor
$$F = \partial \vec{x} / \partial \vec{X}$$
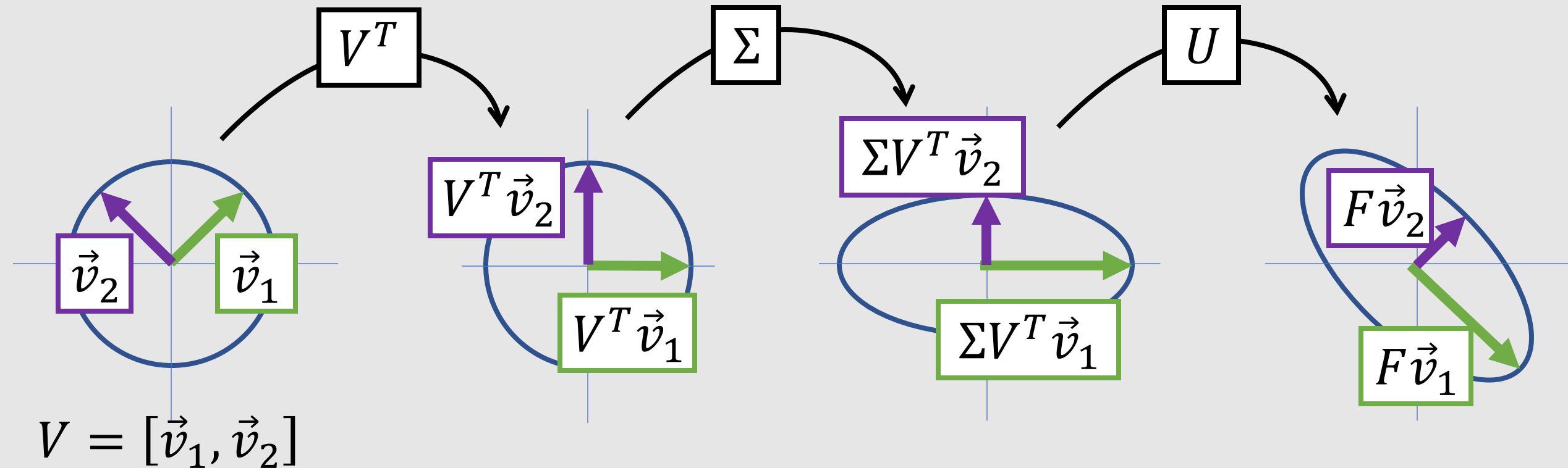
$d\vec{X}$

linear form

$$d\vec{x} = F d\vec{X}$$

SVD of Deformation Gradient Tensor

$$F = U\Sigma V^T \qquad \Sigma = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix}$$

$V = [\vec{v}_1, \vec{v}_2]$

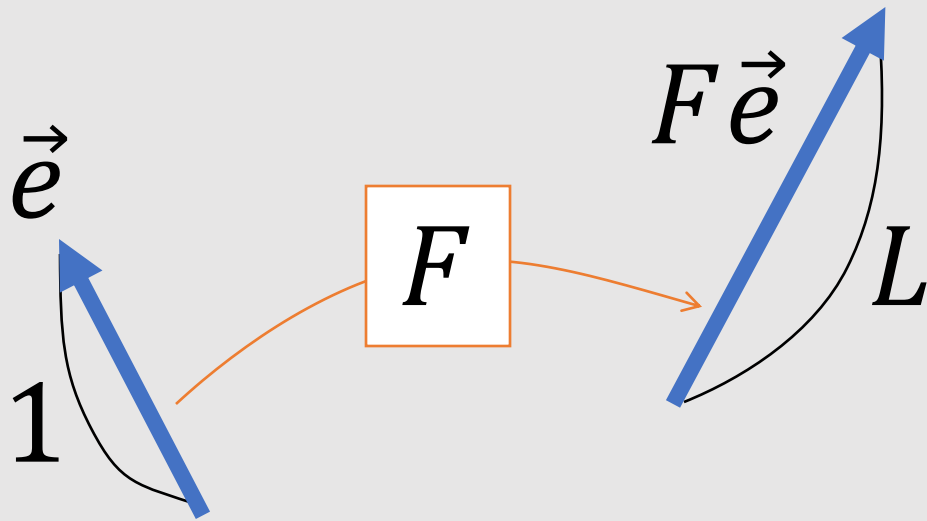# SVD of Deformation Gradient Tensor

$$F = U\Sigma V^T \qquad \Sigma = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix}$$

- Σ is the value we want for energy
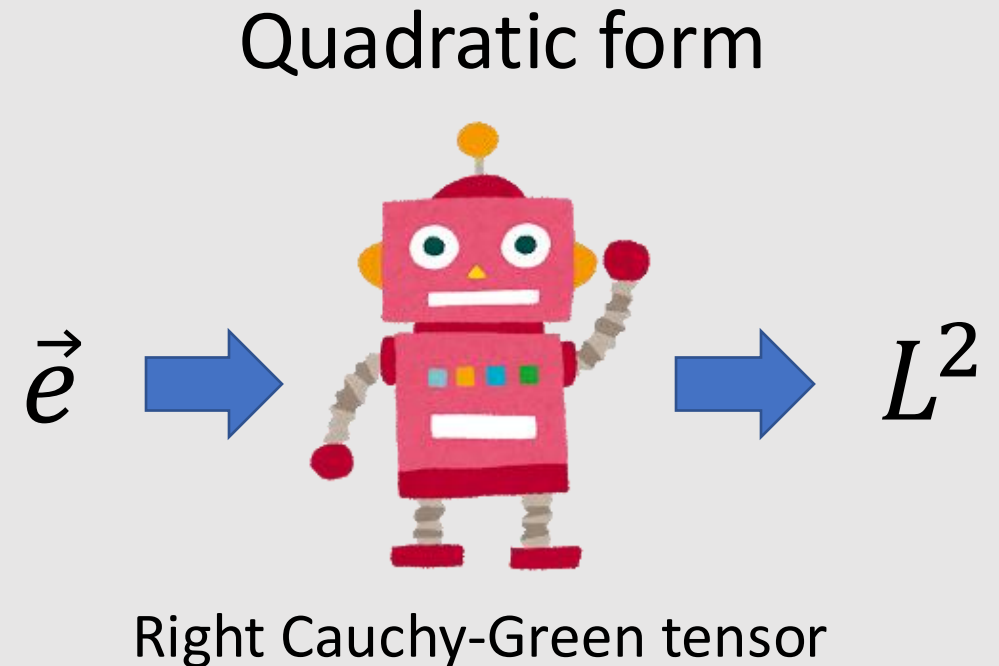- but SVD is costly
- How can we obtain Σ without SVD?

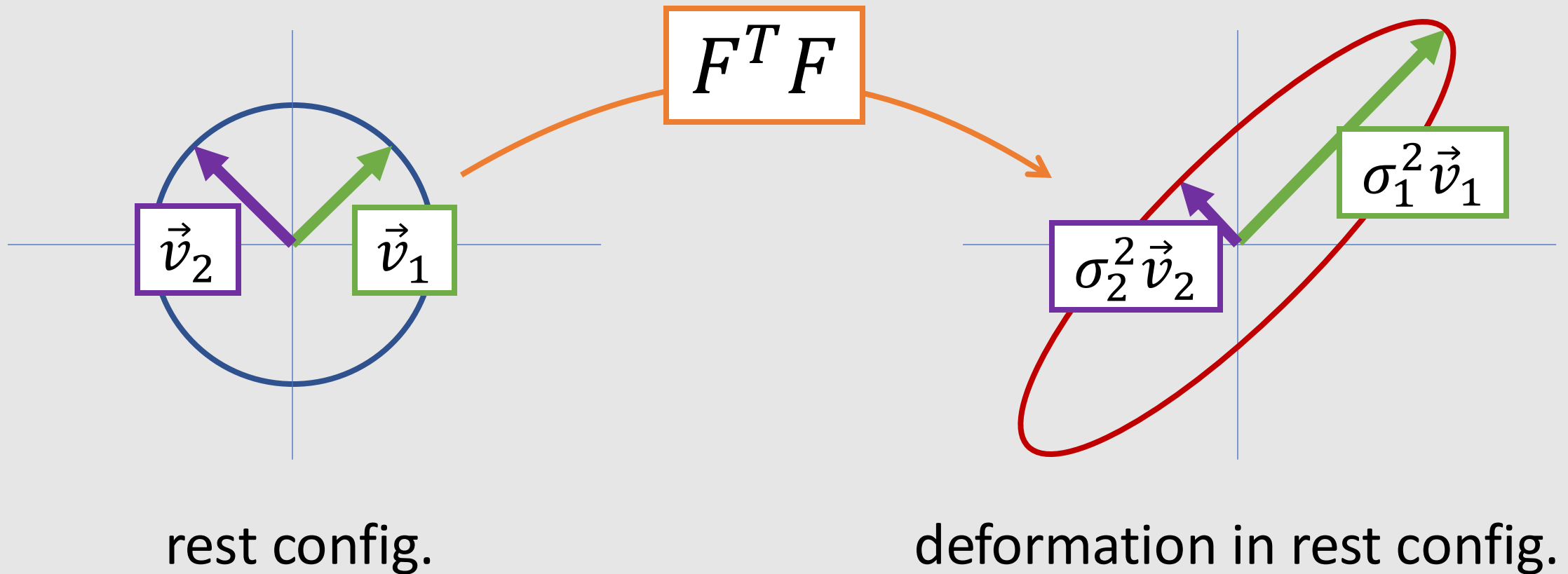# Gram Matrix $F^T F$ Stands for Length Change

- $C = F^T F$: right Cauchy-Green tensor

$$L^2(\vec{e}) = \vec{e}^T F^T F \vec{e}$$
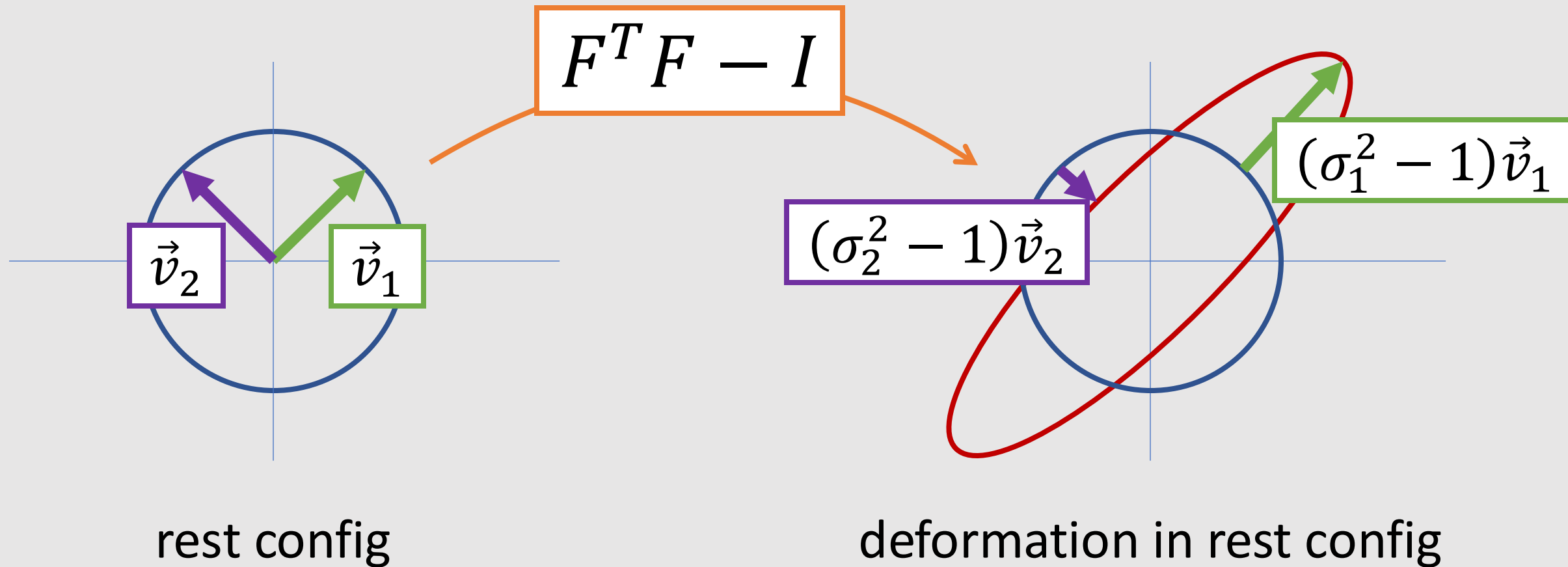
Quadratic form

Right Cauchy-Green tensor

Eigenvalue: Right Cauchy Green Tensor $F^T F$

- Right Cauchy Green tensor is symmetric $F^T F = V\Sigma^2 V^T$

- Eigenvalues of $F^T F$ is squared of singular values : $\sigma_1^2, \sigma_2^2, \sigma_3^2$

$F^T F$

$\vec{v}_2$ $\vec{v}_1$

$\sigma_2^2 \vec{v}_2$ $\sigma_1^2 \vec{v}_1$

rest config.

deformation in rest config.

Eigenvalue: Green Lagrange Tensor $F^T F - I$

$$F^T F - I$$

$\vec{v}_2$

$\vec{v}_1$
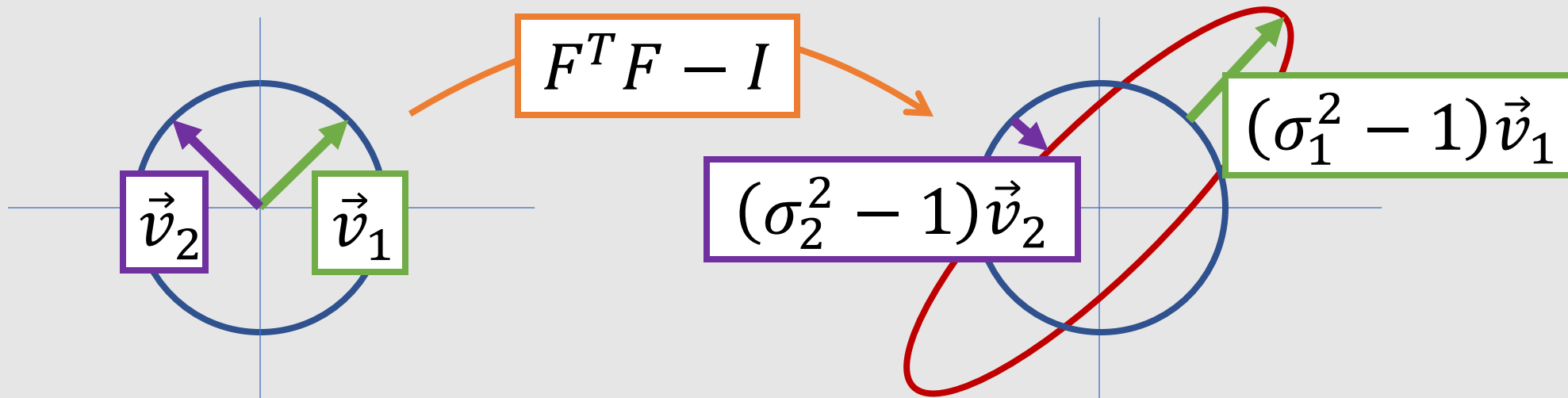
$(\sigma_2^2 - 1)\vec{v}_2$

$(\sigma_1^2 - 1)\vec{v}_1$

rest config

deformation in rest config

Making Energy from Eigenvalue

- Energy for isotropic material

$$W(F) = \|F^T F - I\|_F^2 = (\sigma_1^2 - 1)^2 + (\sigma_2^2 - 1)^2 + (\sigma_3^2 - 1)^2$$

$F^T F - I$

$(\sigma_2^2 - 1)\vec{v}_2$

$(\sigma_1^2 - 1)\vec{v}_1$

$\vec{v}_2$

$\vec{v}_1$

Let's put penalty on the absolute value of $(\sigma_1^2 - 1)$, $(\sigma_2^2 - 1)$ and $(\sigma_3^2 - 1)$

How can We Formulate Elastic Energy?

Strategy A: Elastic energy W is a function of eigenvalues of Green-Lagrange strain $E = F^T F - I$

$$W = \|F^T F - I\|_F^2, \quad \text{where } F = \partial \vec{x} / \partial \vec{X}$$

cancel rotation   *cancel translation*

Hard to choose!

Strategy B: Elastic energy W is a sum of square distances after cancelling rotation and translation

$$W = \min_{R,\vec{t}} \sum_i \omega_i \|R\vec{X}_i + \vec{t} - \vec{x}_i\|^2$$

Making Energy from Eigenvalue

- Energy for isotropic material

$$W(F) = \|F^T F - I\|_F^2 = (\sigma_1^2 - 1)^2 + (\sigma_2^2 - 1)^2 + (\sigma_3^2 - 1)^2$$

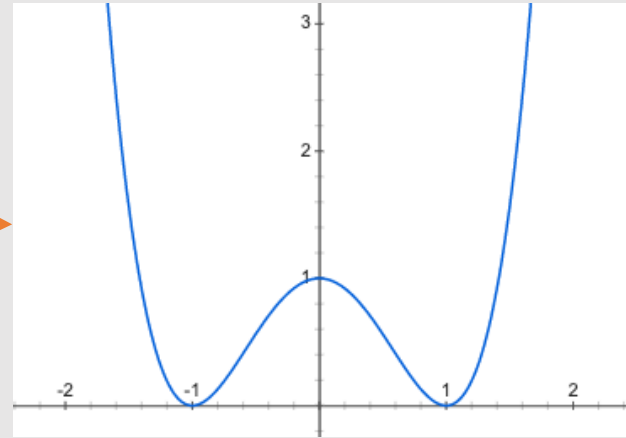This energy doesn't have costly SVD and eigen decomposition easy to compute gradient & hessian!

Making Energy from Eigenvalue

$$W(F) = \|F^T F - I\|_F^2 = (\sigma_1^2 - 1)^2 + (\sigma_2^2 - 1)^2 + (\sigma_3^2 - 1)^2$$

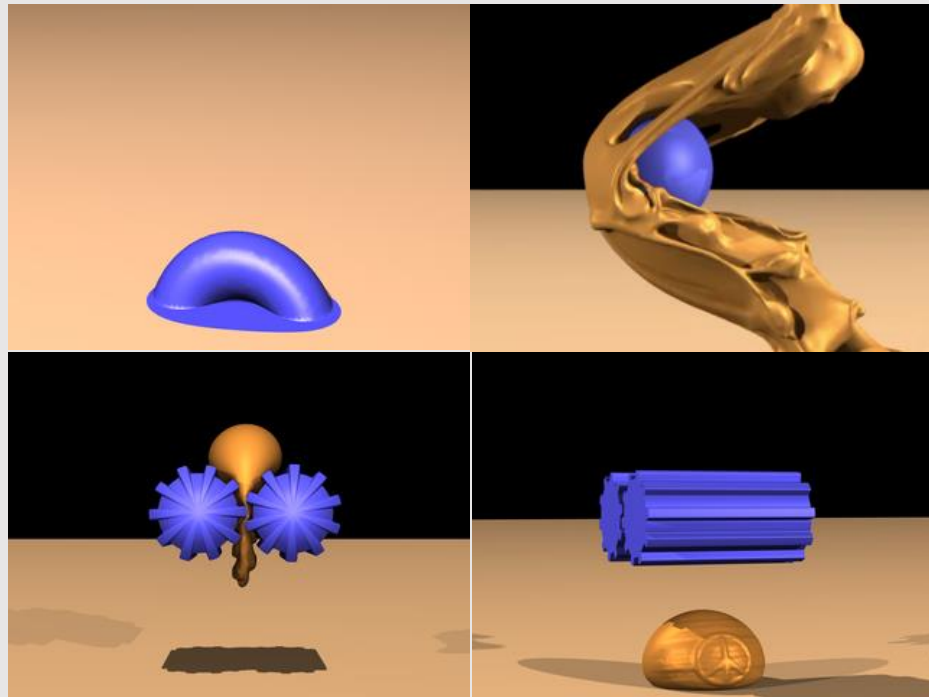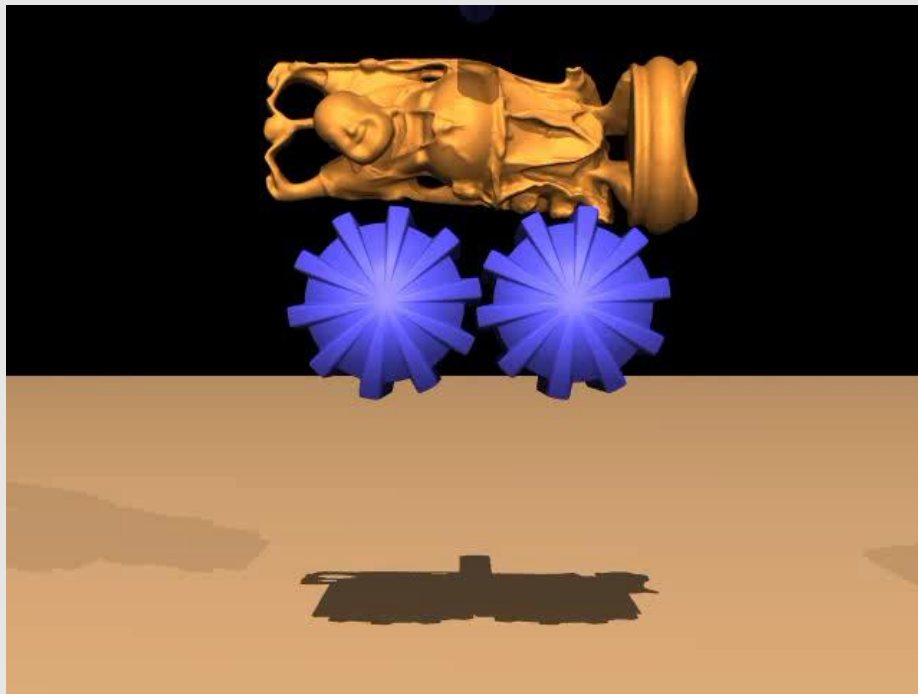Wait... $W(F) = 0$ is not always no deformation. What about mirror reflection $\sigma_i = -1$?

$y = (x^2 - 1)^2$

This energy is not robust to inversion

Invertible FEM [Irving et al. 2004]

- Elastic potential energy based on singular values of F that are σ_i, not on the eigen values of $F^T F$ that are σ_i^2

G. Irving, J. Teran, and R. Fedkiw. 2004. Invertible finite elements for robust simulation of large deformation. In Proceedings of the 2004 ACM SIGGRAPH/Eurographics symposium on Computer animation (SCA '04)