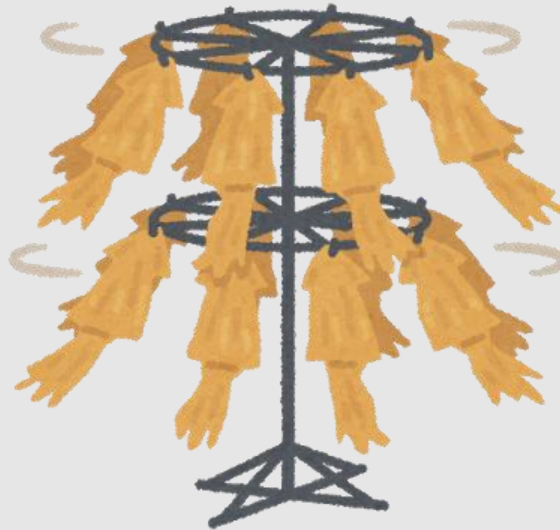


Rotation

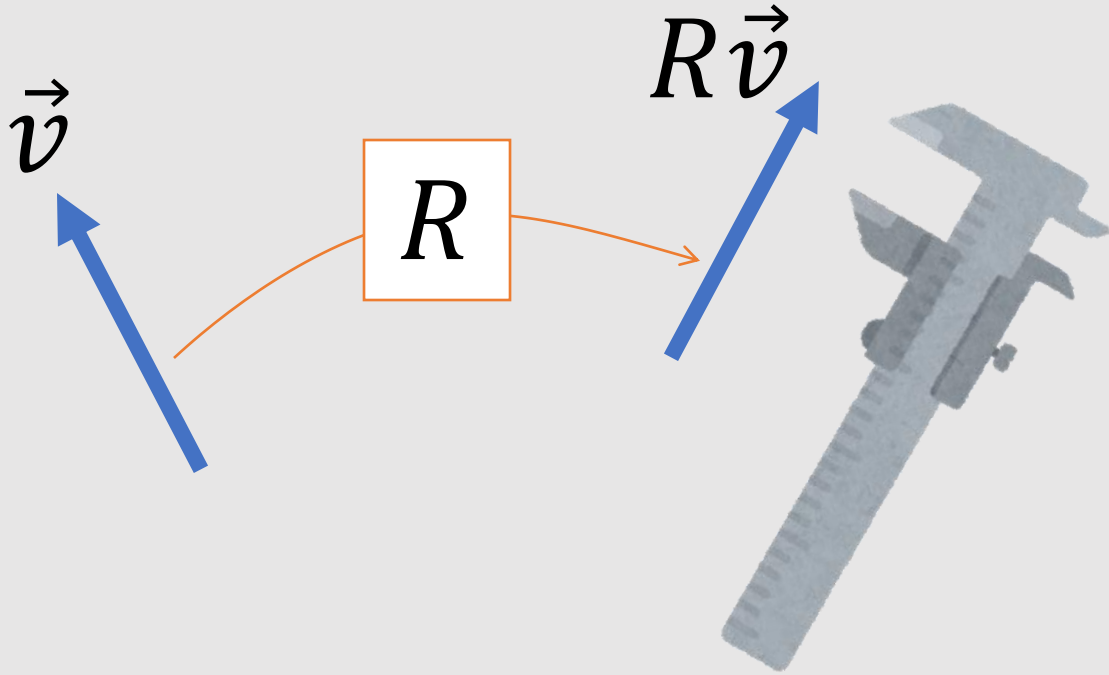
What is Rotation?

- linear transformation R is a rotation if **length does not change** and **volume does not flip**



What is Rotation?

- linear transformation R is a rotation if **length does not change** and volume does not flip



$$\|\vec{v}\|^2 = \|R\vec{v}\|^2$$

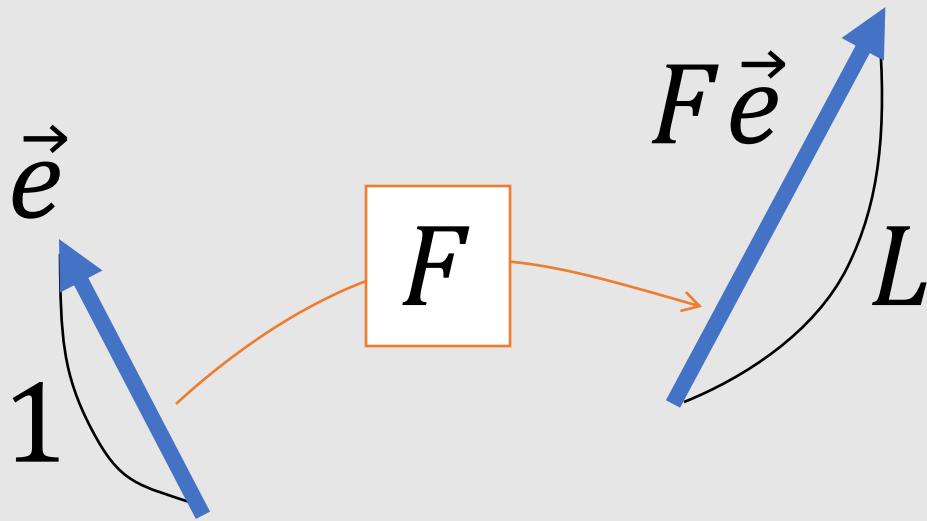
$$\vec{v}^T \vec{v} = \vec{v}^T R^T R \vec{v}$$

$\vec{v}$ can be arbitrary!

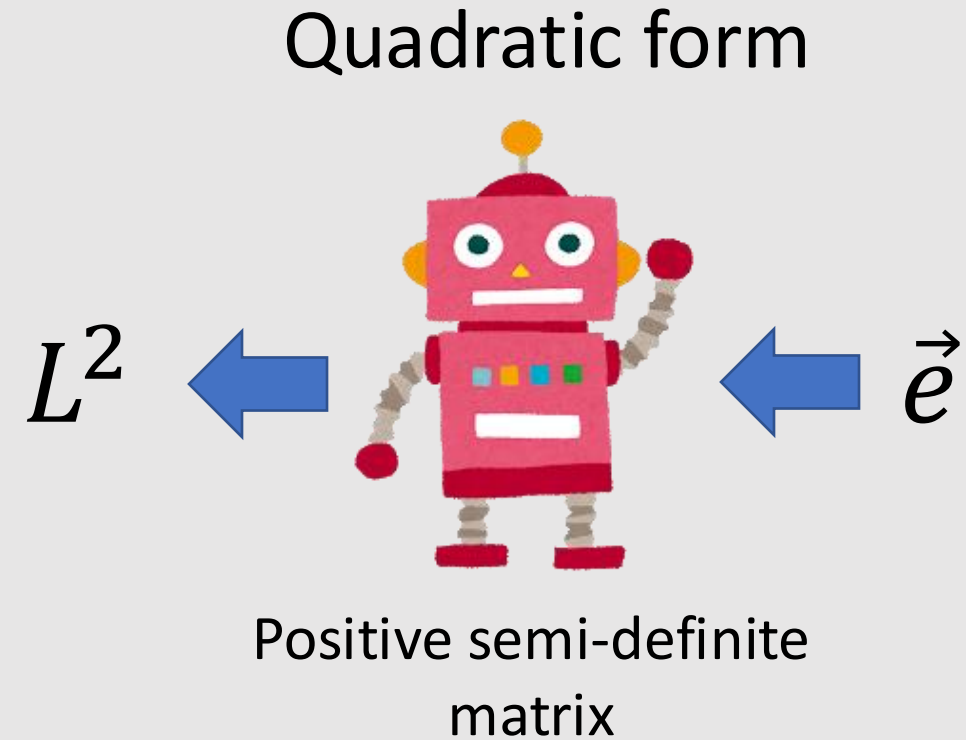
$$R^T R = I$$



Gram Matrix $F^T F$ Stands for Deformed Len.

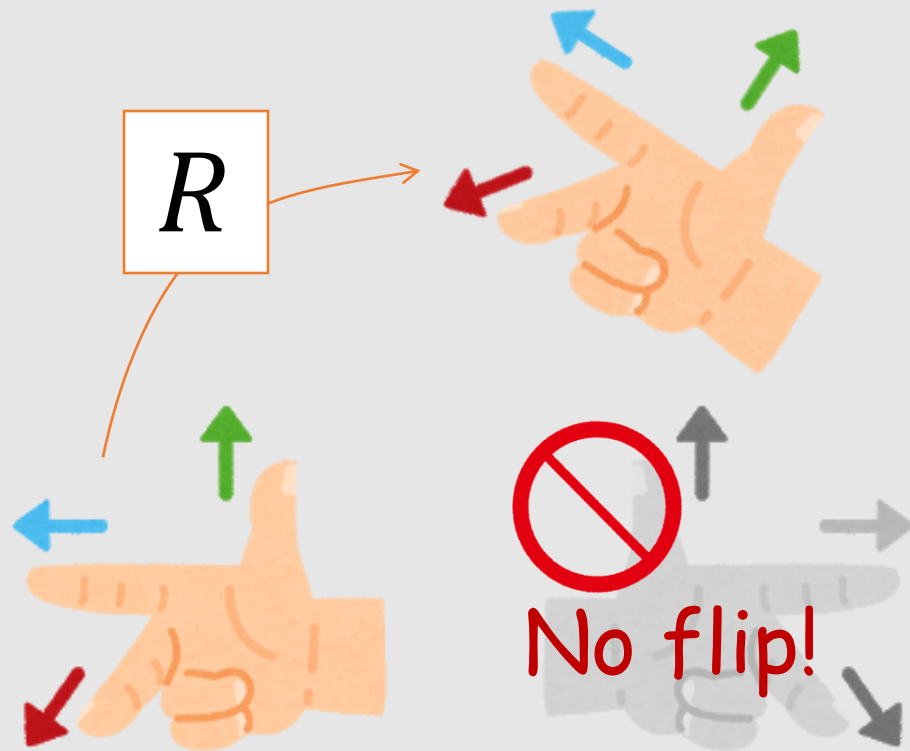


$$L^2(\vec{e}) = \vec{e}^T F^T F \vec{e}$$



What is Rotation?

- linear transformation R is a rotation if length does not change and **volume does not flip**



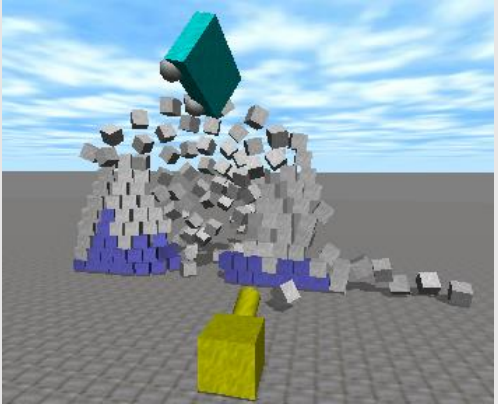
$$\det(R^T R) = \det(I)$$

$$\det(R)^2 = 1$$

$$\det(R) = \pm 1$$

Example of Applications of Rotation

rigid body
animation



Credit: Kborer @Wikipedia

character
animation



Credit: Banlu
Kemiyatorn@Wikipedia

robotics



Credit: Jo
Teichmann@Wikipedia

stereo
reconstruction



Credit: Kbosak@Wikipedia

Representation of 3D Rotation

For large rotation:

- Rotation matrix
- Axis-angle formulation
- Euler angle
- Quaternion



We four are awesome!

For small rotation:

- Infinitesimal rotation

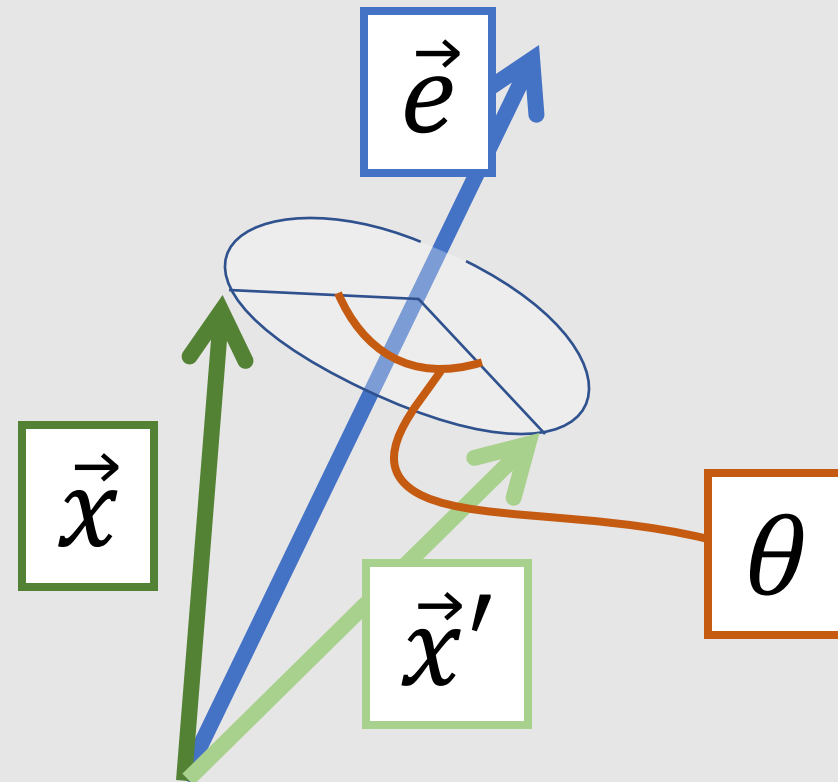


I will be an idol when I grow up!

Rotation Around Axis

- The rotation is parameterized by axis vector $\vec{e}$ and angle θ

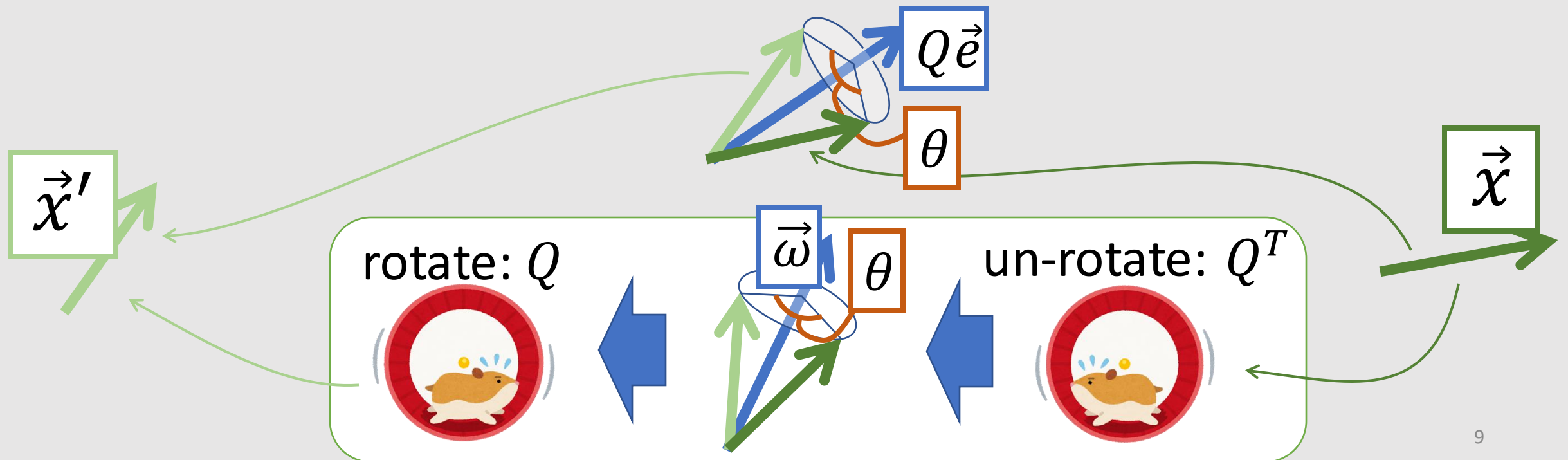
$$\vec{x}' = R(\vec{e}, \theta)\vec{x}$$



Rotation around **Rotated Axis**

- Axis is rotated with $Q \rightarrow$ un-rotate the object with Q^T then rotate around $\vec{\omega}$, then rotate back with Q

$$R(Q\vec{\omega}, \theta) = Q * R(\vec{\omega}, \theta) * Q^T$$





intrinsic rotation:
axis rotated

$$R(Q\vec{e}, \theta) = Q * \boxed{R(\vec{e}, \theta)} * Q^T$$



extrinsic rotation:
axis fix

$$R(Q\vec{e}, \theta_2) * Q = Q * R(\vec{e}, \theta_2)$$

matrix don't commute!



Rotation around Rotated Axis: **Robotic Arm**

- Rotation of end-effector in a 4-link articulated body

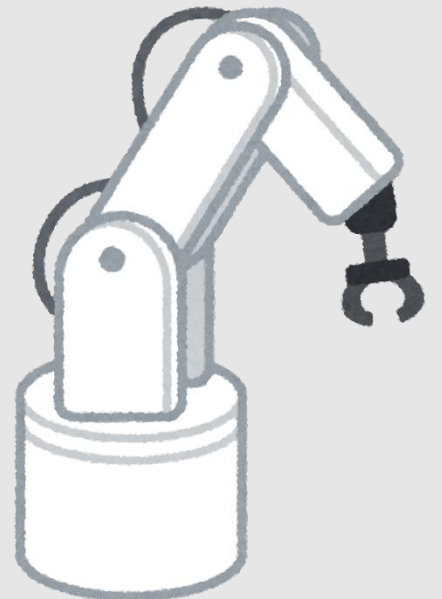
intrinsic

$$R_4(R_3 R_2 R_1 \vec{e}_4, \theta_4) * R_3(R_2 R_1 \vec{e}_3, \theta_3) * R_2(R_1 \vec{e}_2, \theta_2) * R_1(\vec{e}_1, \theta_1)$$



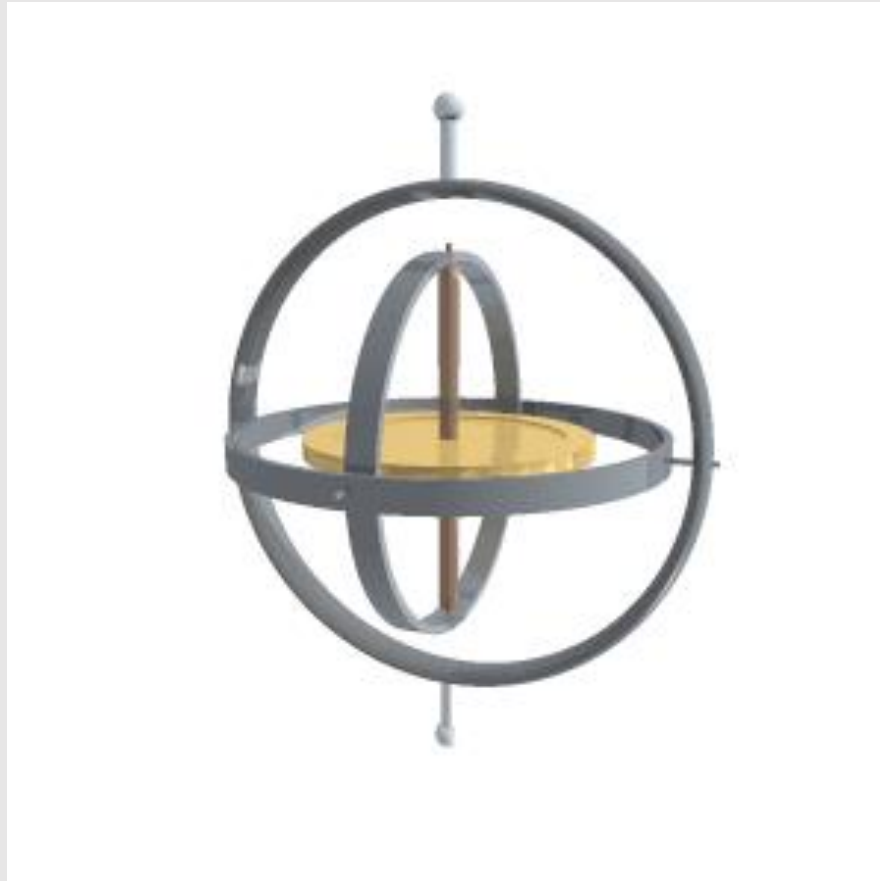
extrinsic

$$R_1(\vec{e}_1, \theta_1) * R_2(\vec{e}_2, \theta_2) * R_3(\vec{e}_3, \theta_3) * R_4(\vec{e}_4, \theta_4)$$



Rotation around Rotated Axis: **Gimbal**

- Gimbal is used to let object freely rotate (e.g, gyroscope)



Credit: Lucas Vieira @ Wikipedia

Euler Angle for Rotation Parameterization

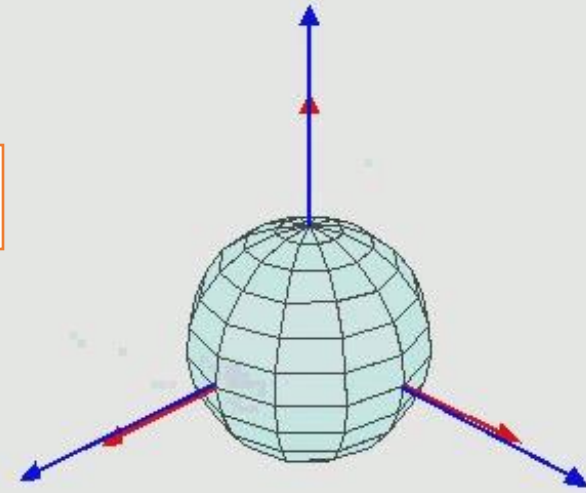
intrinsic

$$\vec{x}' = R_3(R_2 R_1 \vec{e}_Z, \theta_3) * R_2(R_1 \vec{e}_X, \theta_2) * R_1(\vec{e}_Z, \theta_1) \vec{x}$$

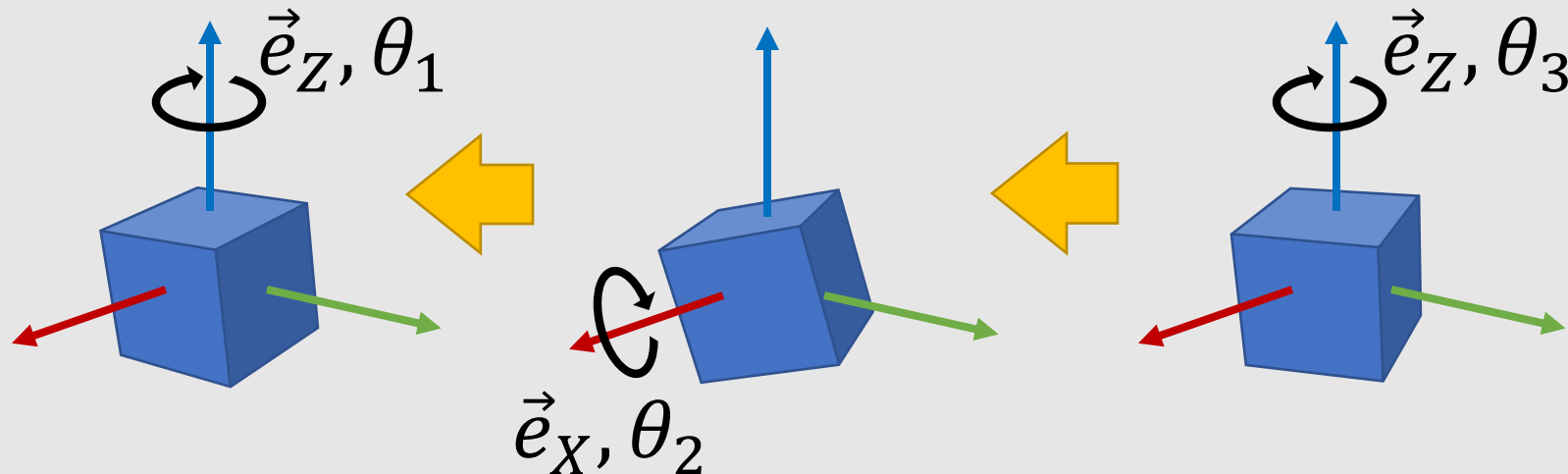


extrinsic

$$\vec{x}' = R_1(\vec{e}_Z, \theta_1) * R_2(\vec{e}_X, \theta_2) * R_3(\vec{e}_Z, \theta_3) \vec{x}$$



credit: Xavax @ Wikipedia



Vector Rotation Parameterizations

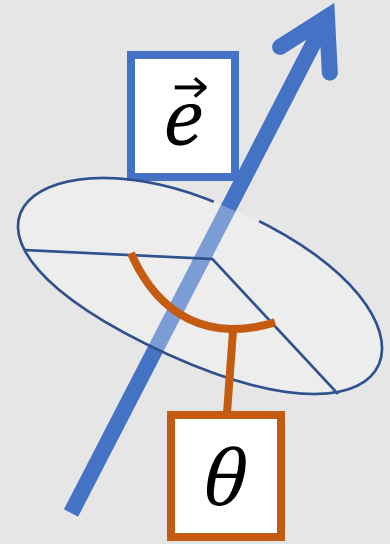
Euler vector (rotation vector): $\vec{\omega} = \vec{e}\theta$

Scaling unit axis vector $\vec{e}$ with trigonometric func. of θ

- 360° rotation become zero rotation
- Elegantly composite rotations

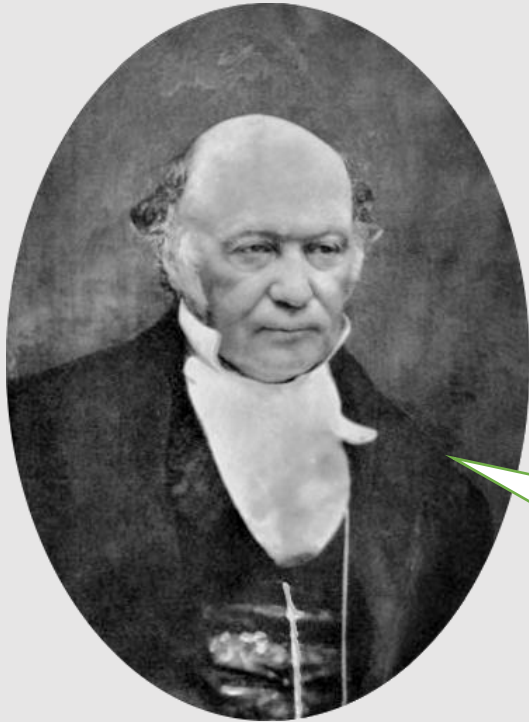
Cayley-Gibbs-Rodrigues parameterization: $\vec{e} \tan \frac{\theta}{2}$

Euler-Rodrigues parameterization: $\left(\vec{e} \sin \frac{\theta}{2}, \cos \frac{\theta}{2} \right)$



Discovery of Quaternion

- Sir William Rowan Hamilton (1805–1865) from Ireland



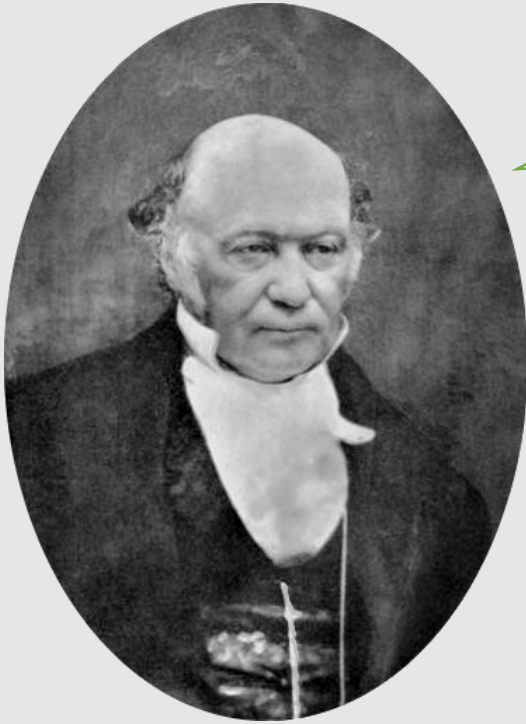
Pa, imaginary number
for 3D yet?



No clue for 10 years... 🤔
Let me take a walk for a change...

Discovery of Quaternion

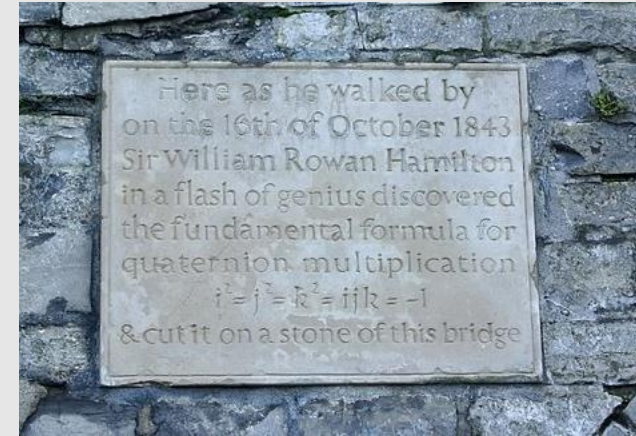
- Sir William Rowan Hamilton (1805–1865) from Ireland



Finally got an idea 🎉
 $i^2 = j^2 = k^2 = ijk = -1$



Credit: Wisher@wikipedia



Credit: Cone83@wikipedia

Properties of Quaternion

- One real number and three imaginary numbers

$$q = r + v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$$

- Conjugate quaternion

$$\bar{q} = r - v_x \mathbf{i} - v_y \mathbf{j} - v_z \mathbf{k}$$

- Norm of quaternion

$$\|q\|^2 = q\bar{q} = r^2 + v_x^2 + v_y^2 + v_z^2$$

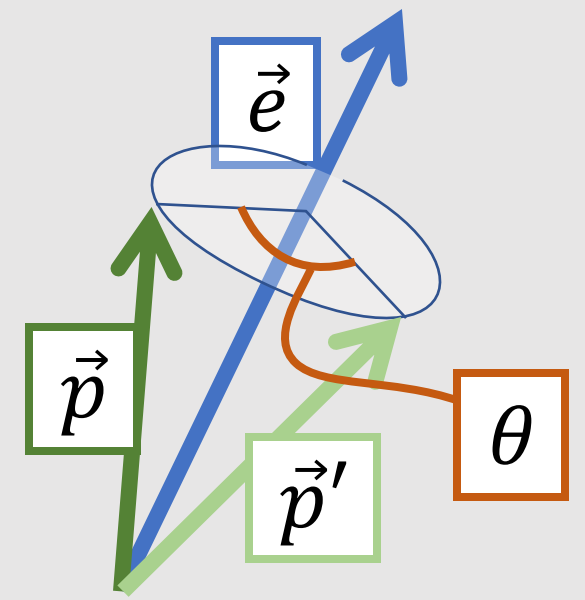
Rotation with Quaternion

Euler-Rodrigues
parameterization

$$q = \cos \frac{\theta}{2} + \sin \frac{\theta}{2} (e_x \mathbf{i} + e_y \mathbf{j} + e_z \mathbf{k})$$

$$p = p_x \mathbf{i} + p_y \mathbf{j} + p_z \mathbf{k}$$

$$p' = p'_x \mathbf{i} + p'_y \mathbf{j} + p'_z \mathbf{k}$$



input vector

$$p' = q p \bar{q}$$

rotated vector

Composite Rotation

- Multiplication of quaternion is associative

rotation with q_1 and then q_2


$$q_{12} = q_2 q_1$$

quaternion don't commute!



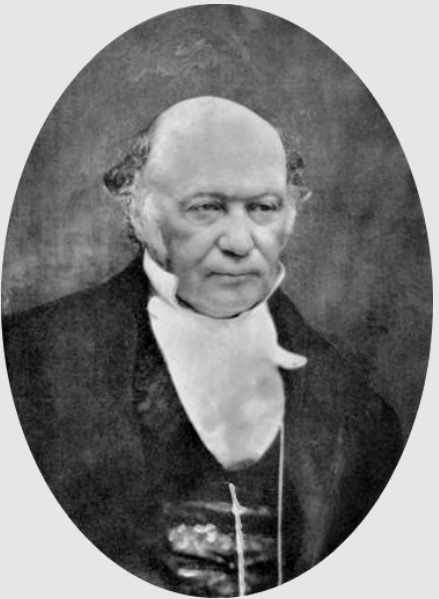
Cayley–Hamilton Theorem

- Eigen values are the root of the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = \sum c_i \lambda^i = 0$$

Characteristic polynomial of a matrix A produces zero ,
when inputting A

$$p(A) = \sum c_i A^i = 0$$



Cayley–Hamilton Theorem for a **3x3 Matrix**

- Characteristic polynomial of a matrix A produces zero , when inputting A

$$p(A) = A^3 + c_1 A^2 + c_2 A + c_3 I = 0$$

$$c_1 = \text{tr}(A) = \lambda_1 + \lambda_2 + \lambda_3$$

$$c_2 = 1/2 \{ \text{tr}(A)^2 - \text{tr}(A^2) \} = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1$$

$$c_3 = \det(A) = \lambda_1 \lambda_2 \lambda_3$$

Infinitesimal Rotation is a Vector

- What if the rotation is very small?

$$R = I + \Omega$$



$$R^T R = (I + \Omega)^T (I + \Omega) = I$$



$$\Omega^T = -\Omega$$

Ω is a **skew**
symmetric matrix!



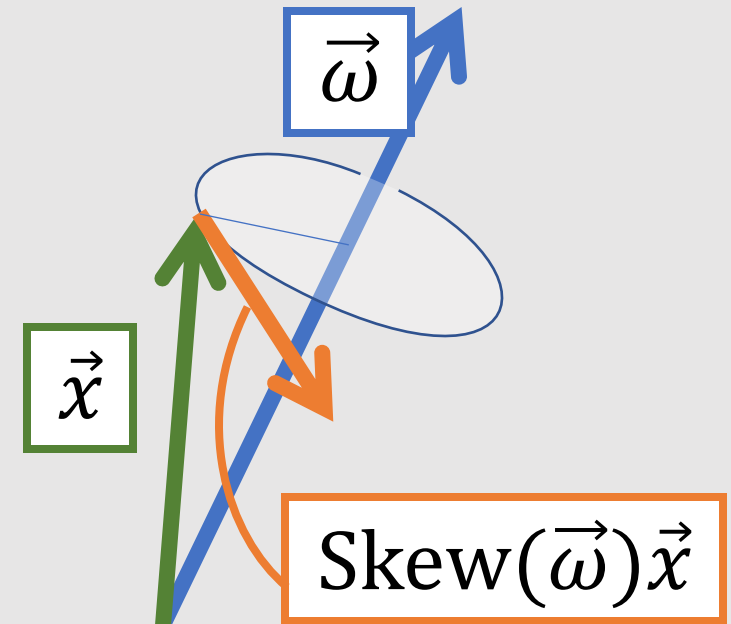
Properties of 3x3 Skew Symmetric Matrix

- Skew matrix has only three independent elements

$$\Omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \equiv \text{Skew}(\vec{\omega})$$

- Skew matrix defines cross product

$$\Omega \vec{v} = \vec{\omega} \times \vec{v} \quad \vec{\omega} = (\omega_1, \omega_2, \omega_3)^T$$



3x3 Skew Symmetric Matrix **Squared**

$$\begin{aligned}\Omega^2 = \text{Skew}(\vec{\omega})^2 &= \begin{bmatrix} -\omega_2^2 - \omega_3^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & -\omega_3^2 - \omega_1^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & -\omega_1^2 - \omega_2^2 \end{bmatrix} \\ &= \vec{\omega} \otimes \vec{\omega} - \vec{\omega}^T \vec{\omega} I\end{aligned}$$

$$\begin{aligned}\text{tr}(\Omega^2) &= \omega_1^2 + \omega_2^2 + \omega_3^2 - 3(\omega_1^2 + \omega_2^2 + \omega_3^2) \\ &= -2\|\vec{\omega}\|^2\end{aligned}$$

3x3 Skew Symmetric Matrix **Cubed**

Cayley-Hamilton's theorem

$$p(\Omega) = \Omega^3 + (\text{tr}\Omega)\Omega^2 + \frac{(\text{tr}\Omega)^2 - \text{tr}(\Omega^2)}{2}\Omega + (\det\Omega)I = 0$$



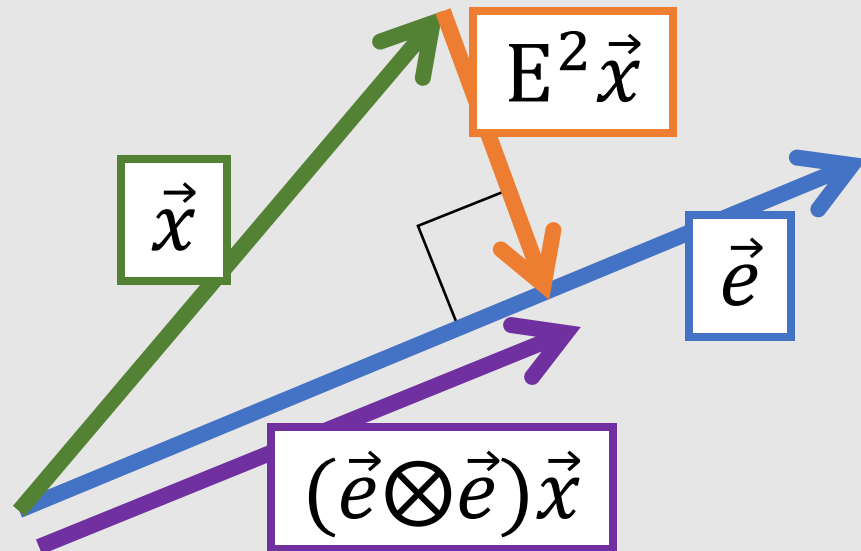
$$\text{tr}\Omega = 0, \det\Omega = 0, \text{tr}(\Omega^2) = -2\|\vec{\omega}\|^2$$

$$\Omega^3 = -\|\vec{\omega}\|^2\Omega$$

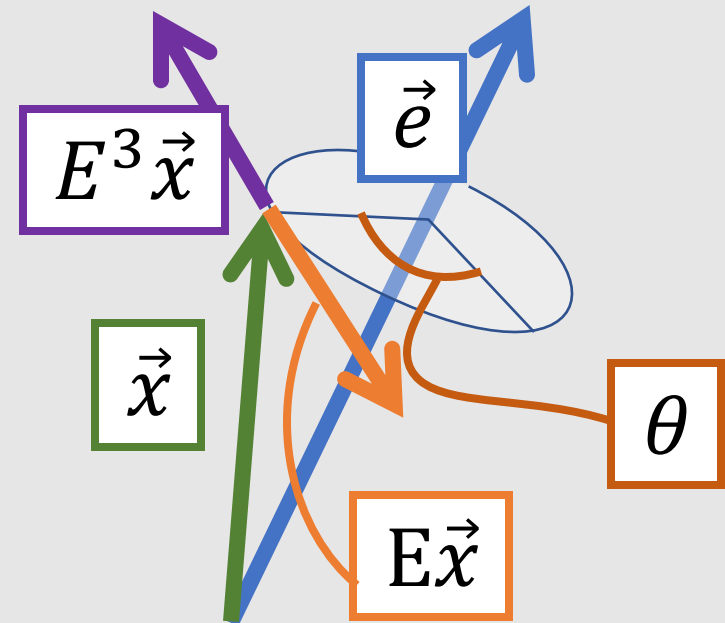
Skew Symmetric Matrix of **Unit Vector** $\vec{e}$

$$E = \text{Skew}(\vec{e}), \quad \text{where } \|\vec{e}\| = 1$$

$$E^2 = \vec{e} \otimes \vec{e} - I$$



$$E^3 = -E$$



From Infinitesimal Rotation Vector to Matrix

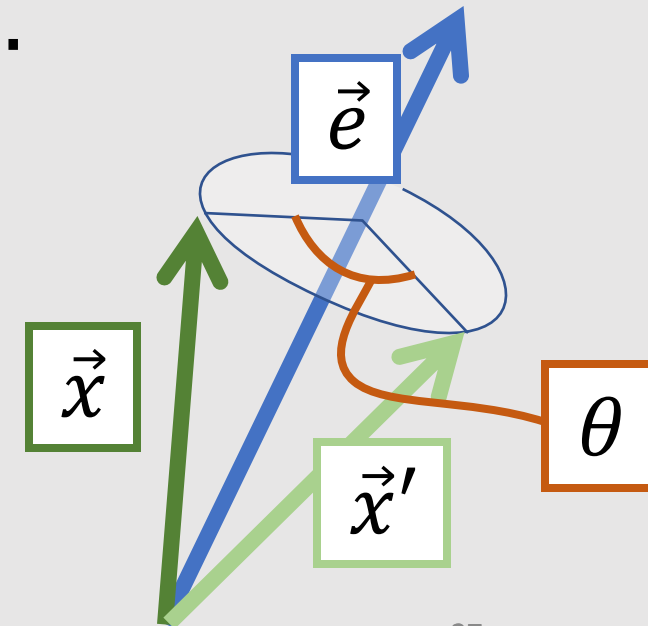
$$R(\vec{e}, \theta) = \exp(\theta E)$$

$$E = \text{Skew}(\vec{e}) = \begin{bmatrix} 0 & e_3 & -e_2 \\ -e_3 & 0 & e_1 \\ e_2 & -e_1 & 0 \end{bmatrix}$$

$$= I + \frac{\theta}{1!} E + \frac{\theta^2}{2!} E^2 + \frac{\theta^3}{3!} E^3 + \frac{\theta^4}{4!} E^4 \dots$$

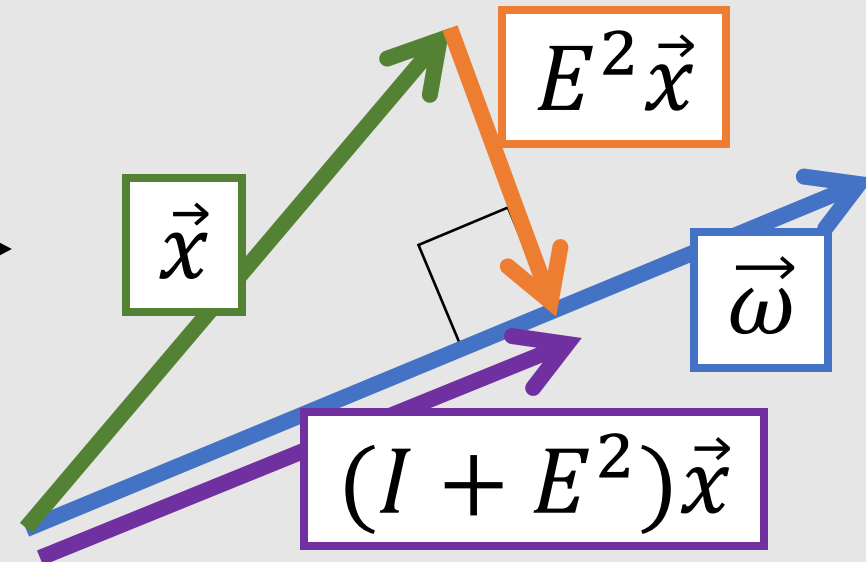
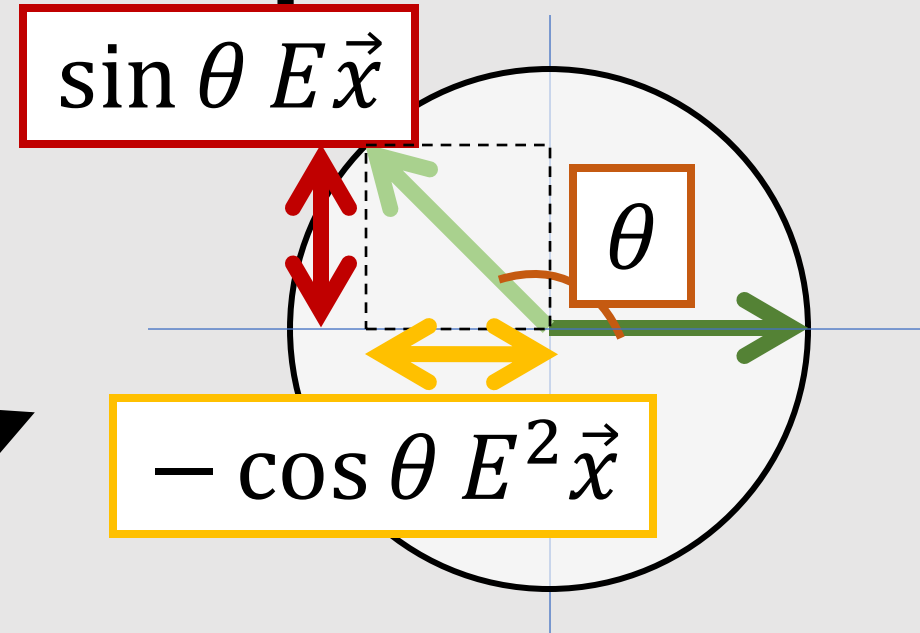
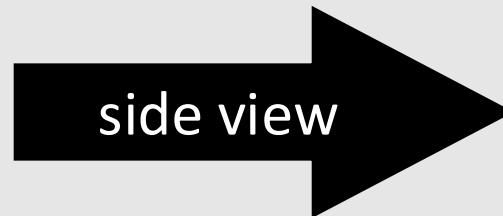
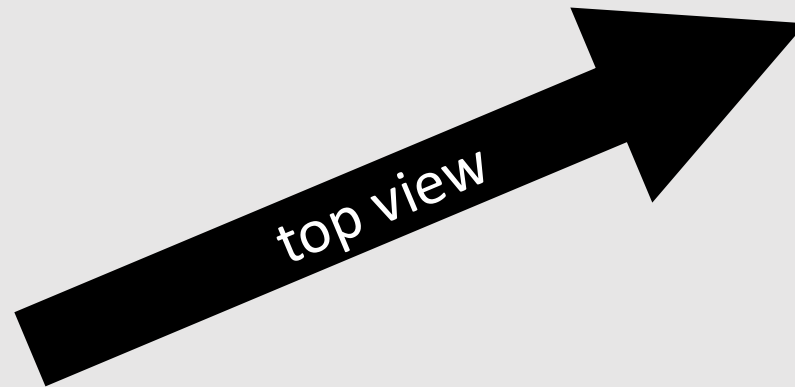
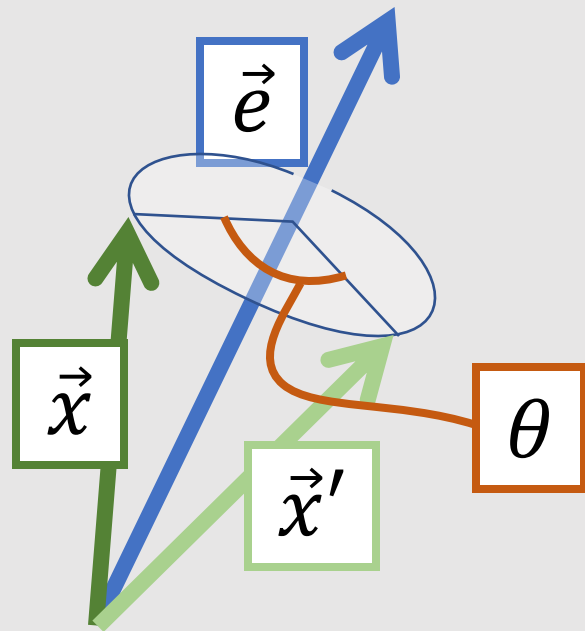
$$= I + \sin\theta E + (1 - \cos\theta) E^2$$

Rodriguez's rotation formula



Rodriguez's Rotation Formula Explained

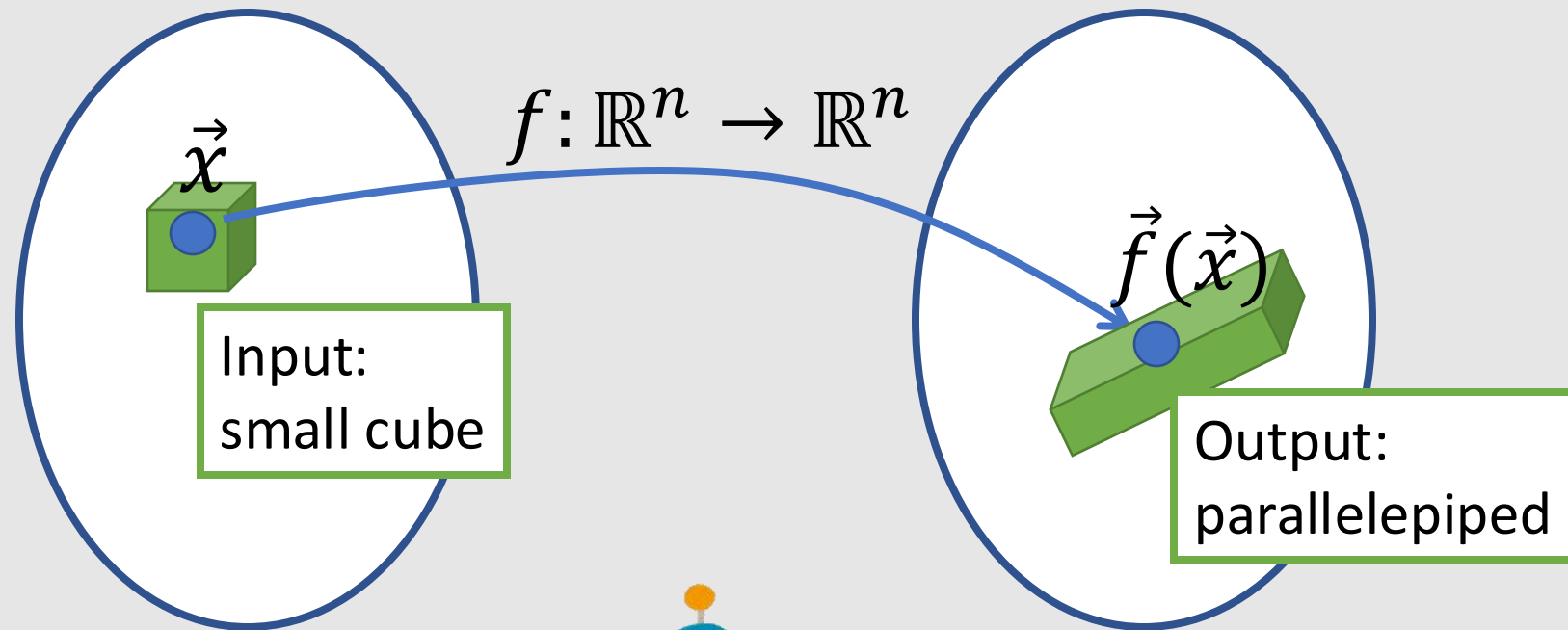
$$R(\vec{e}, \theta) = I + \sin\theta E + (1 - \cos\theta)E^2$$



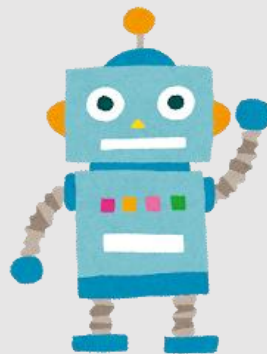
Jacobian Determinant: Volume Change Ratio

input space $\mathbb{R}^n$

output space $\mathbb{R}^n$



Input
volume: dv



Output
volume = $\det(J) dv$

$$\text{Jacobian } J = \nabla \vec{f}$$

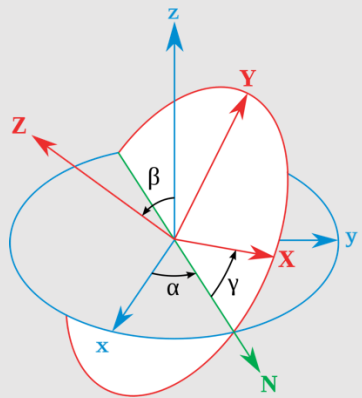
Jacobian Determinant: Volume Change Ratio

input:

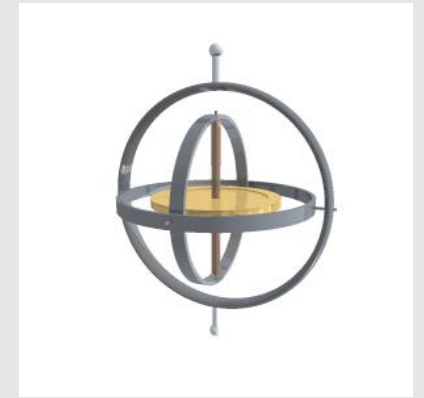
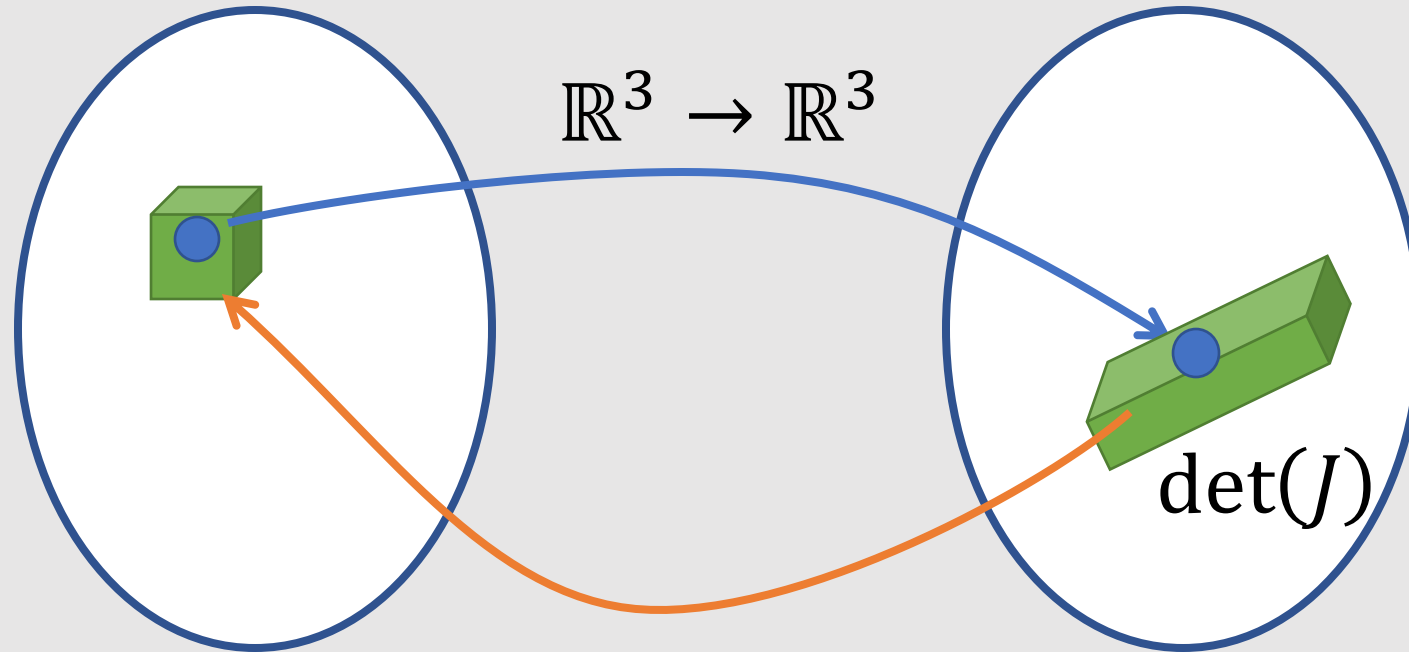
small change in Euler angle
 $(d\theta_1, d\theta_2, d\theta_3)$

output:

infinitesimal rotation
 $d\vec{\omega} = (d\omega_1, d\omega_2, d\omega_3)$



Credit: Lionel Brits
@ Wikipedia



Credit: Lucas Vieira
@Wikipedia

If output volume is not zero $\det(J) \neq 0$, there is an inverse map

Gimbal Lock: Zero Jacobian Determinant

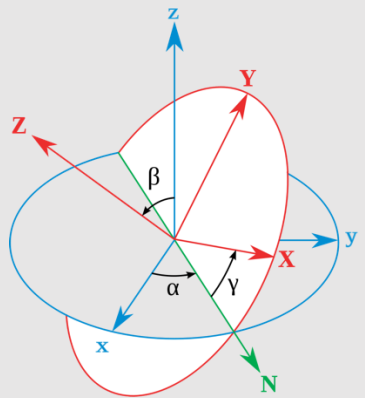
input:

small change in Euler angle
 $(d\theta_1, d\theta_2, d\theta_3)$

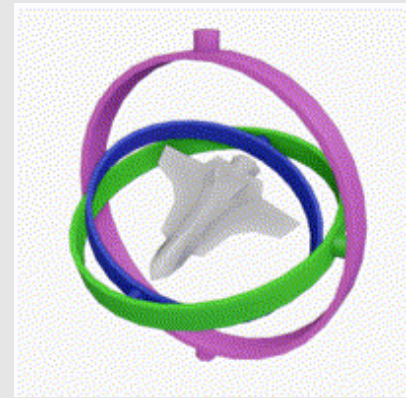
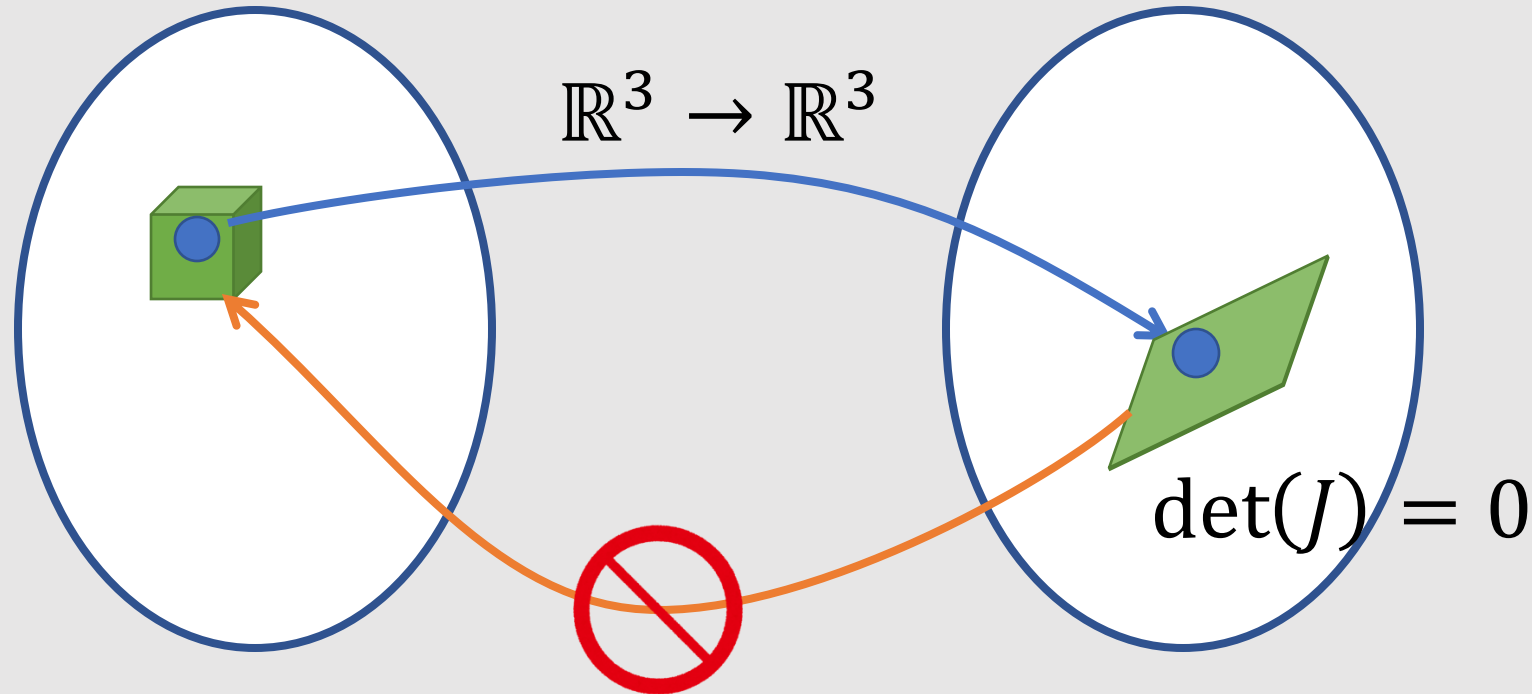
output:

infinitesimal rotation

$$d\vec{\omega} = (d\omega_1, d\omega_2, d\omega_3)$$



Credit: Lionel Brits
@ Wikipedia



Credit: Drummyfish
@ Wikipedia

no inverse map when axes aligned!

Comparison of 3D Rotation Representations

Rotation matrix

- DoF: $3 \times 3 = 9$
- rotation, scale, sheer, mirror
- 😊 general
- ☹️ large

Euler angle

- DoF: 3
- rotation
- 😊 understandable
- ☹️ gimbal lock

Quaternion

- DoF: $1 + 3 = 4$
- rotation, scale
- 😊 compact
- ☹️ not understandable

Axis-angle formulation

Infinitesimal rotation

Rodriguez's formula

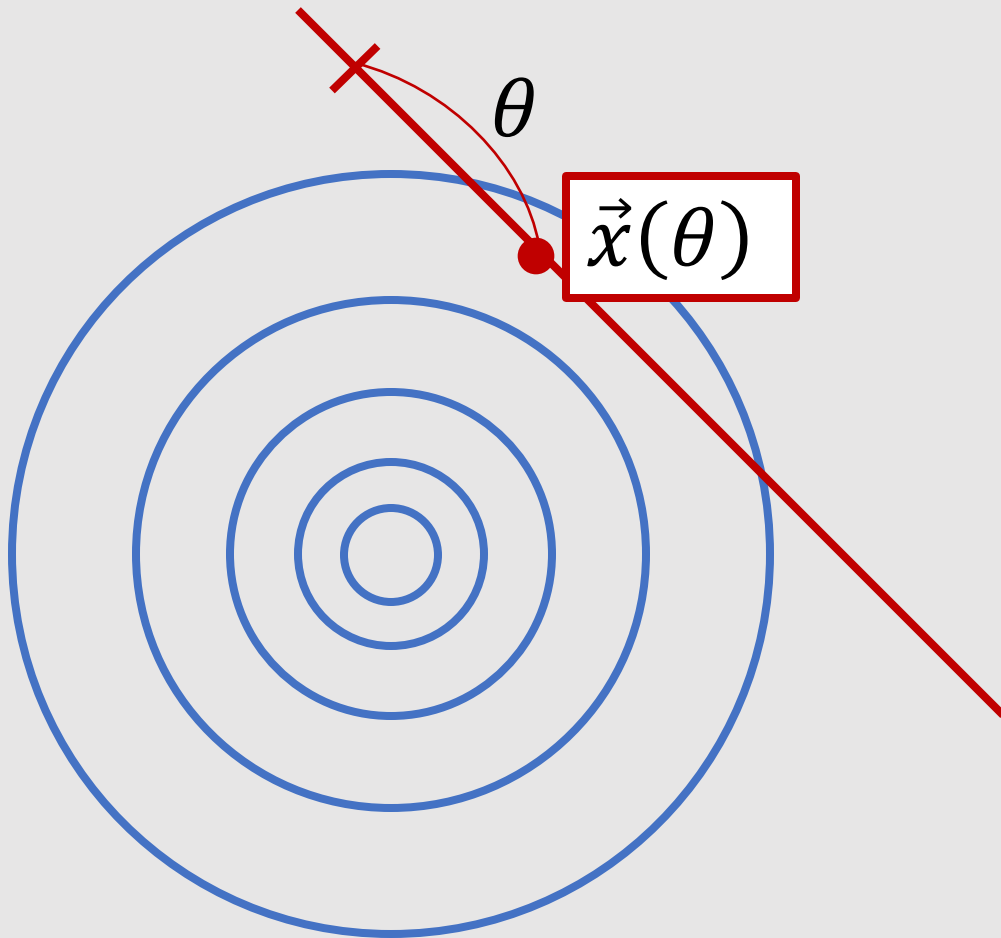
exponential

$$\cos \frac{\theta}{2}, \sin \frac{\theta}{2}$$

differentiation

DoF Elimination using Parameterization

Parameterize solution $\vec{x}(\theta)$ such that constraints naturally satisfy



$$\operatorname{argmin}_{\vec{x} \in \{\vec{x} | g(\vec{x}) = 0\}} W(\vec{x}) \quad \longrightarrow \quad \operatorname{argmin}_{\theta} W(\vec{x}(\theta))$$

e.g., $g(\vec{x}) = x + y + 2 = 0$

$$\longrightarrow \begin{cases} x = +\theta - 1 \\ y = -\theta - 1 \end{cases}$$

Minimize Parameterized Solution

$$\underset{\theta}{\operatorname{argmin}} W(\vec{x}(\theta))$$

Newton-Raphson method

$$d\theta = - \left[\frac{\partial^2 W}{\partial \theta^2} \right]^{-1} \left(\frac{\partial W}{\partial \theta} \right)$$

find the **root** of gradient!



Differentiation of Rotation Matrix

Change under constraints $R(t)^T R(t) = I \rightarrow$ DoF elimination

walking on the edge!



$R(t)$

$$R(t)^T R(t) = I$$



Parameterization of Rotation

Let's walk on the line to find the minimum

$$R(t)^T R(t) = I$$

R

$$R \exp\{\text{Skew}(\vec{\omega})\}$$

$$\exp\{\text{Skew}(\vec{\omega})\}$$

Optimization of Rotation: Pattern R^*dR

$$R = R_i \exp\{\text{Skew}(\vec{\omega})\}$$

Minimize $W(R)$

Minimize $W(R(\vec{\omega}))$

Newton method

$$\vec{\omega} = - \left[\frac{\partial^2 W}{\partial^2 \vec{\omega}} \right]^{-1} \left(\frac{\partial W}{\partial \vec{\omega}} \right)$$

$$R_{i+1} = R_i \exp\{\text{Skew}(\vec{\omega})\}$$

$$\begin{aligned} \exp\{\text{Skew}(\vec{\omega})\} \\ = I + \text{Skew}(\vec{\omega}) + \frac{1}{2} \text{Skew}(\vec{\omega})^2 \dots \end{aligned}$$

find the **root** of gradient!



Optimization of Rotation: Pattern dR^*R

$$R = \exp\{\text{Skew}(\vec{\omega})\} R_i$$

Minimize $W(R)$

Minimize $W(R(\vec{\omega}))$

Newton method

$$\vec{\omega} = - \left[\frac{\partial^2 W}{\partial^2 \vec{\omega}} \right]^{-1} \left(\frac{\partial W}{\partial \vec{\omega}} \right)$$

$$R_{i+1} = \exp\{\text{Skew}(\vec{\omega})\} R_i$$

$$\begin{aligned} \exp\{\text{Skew}(\vec{\omega})\} \\ = I + \text{Skew}(\vec{\omega}) + \frac{1}{2} \text{Skew}(\vec{\omega})^2 \dots \end{aligned}$$

find the **root** of gradient!



Gradient and Hessian Including Rotation

$$W(R) = \|R\vec{p} - \vec{q}\|^2$$

the only changing term

$$= (R\vec{p} - \vec{q})^T (R\vec{p} - \vec{q}) = \|\vec{p}\|^2 - 2\vec{q}^T R\vec{p} + \|\vec{q}\|^2$$

$$-2\vec{q}^T R(\vec{\omega})\vec{p} = -2\vec{q}^T \exp\{\text{Skew}(\vec{\omega})\} R\vec{p}$$

$$\cong -2\vec{q}^T \left\{ I + \underbrace{\text{Skew}(\vec{\omega})}_{1^{\text{st}} \text{ order}} + \underbrace{\frac{1}{2} \text{Skew}(\vec{\omega})^2}_{2^{\text{nd}} \text{ order}} \right\} R\vec{p}$$

$$-2\vec{q}^T \text{Skew}(\vec{\omega}) R\vec{p} = -2\vec{\omega}^T \text{Skew}(R\vec{p}) \vec{q} = \vec{\omega}^T (2\vec{q} \times R\vec{p})$$

$$-\vec{q}^T \text{Skew}(\vec{\omega})^2 R\vec{p} = -\vec{q}^T [\vec{\omega} \otimes \vec{\omega} - \vec{\omega}^T \vec{\omega} I] R\vec{p} = \vec{\omega}^T [\vec{q}^T R\vec{p} I - \vec{q} \otimes R\vec{p}] \vec{\omega}$$

Gradient and Hessian Including Rotation

$$W(R(\vec{\omega})) = \|R(\vec{\omega})\vec{p} - \vec{q}\|^2$$

Gradient: $\frac{\partial W}{\partial \vec{\omega}} = \frac{\partial}{\partial \vec{\omega}} \{ \vec{\omega}^T (2\vec{q} \times R\vec{p}) \} = 2\vec{q} \times R\vec{p}$

Hessian:
$$\begin{aligned} \frac{\partial^2 W}{\partial \vec{\omega}^2} &= \frac{\partial^2}{\partial \vec{\omega}^2} \{ \vec{\omega}^T (\vec{q}^T R\vec{p}I - \vec{q} \otimes R\vec{p}) \vec{\omega} \} \\ &= 2\vec{q}^T R\vec{p}I - \vec{q} \otimes R\vec{p} - R\vec{p} \otimes \vec{q} \end{aligned}$$

Hessian must be symmetric!

Differentiation w.r.t Vectors

- Transform the equation into a polynomial

$$\begin{aligned} W(\vec{\omega}) &= \dots \\ &= \dots \\ &= a + \vec{b}^T \vec{\omega} + \vec{\omega}^T C \vec{\omega} + \dots \end{aligned}$$

$$\text{Gradient: } \frac{\partial W}{\partial \vec{\omega}} = \vec{b}$$

$$\text{Hessian: } \frac{\partial^2 W}{\partial \vec{\omega}^2} = C^T + C$$