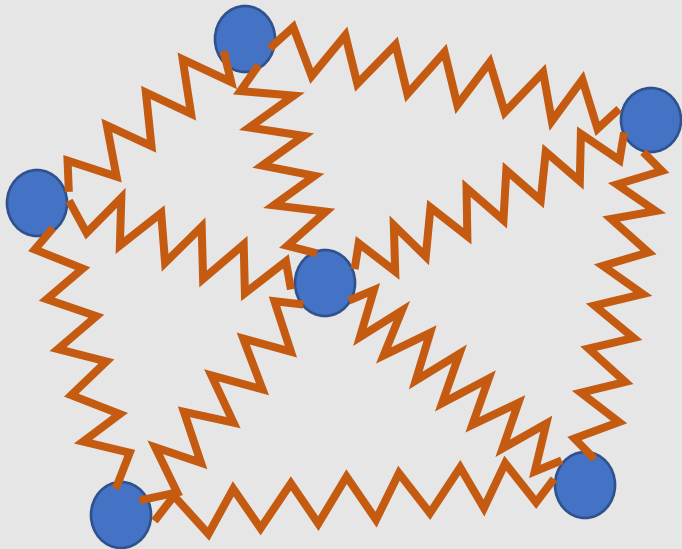


Rigid Body Approximation

剛体近似

Rigid Body Approximation

- In mass-spring system, 5 points in 3D has 15 degrees of freedom

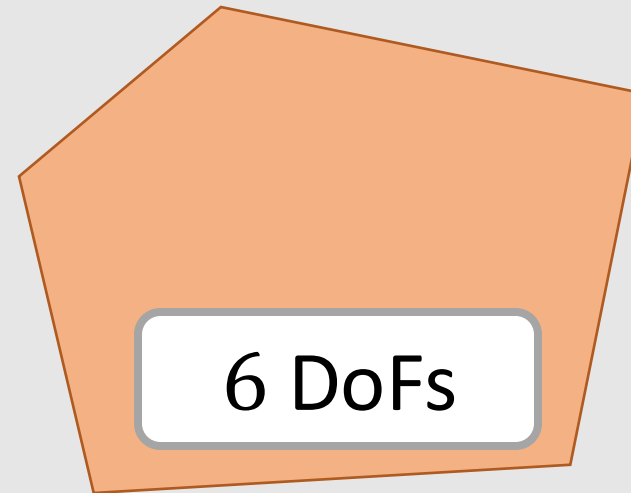
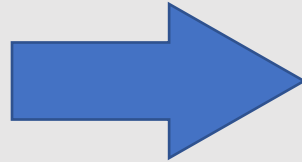
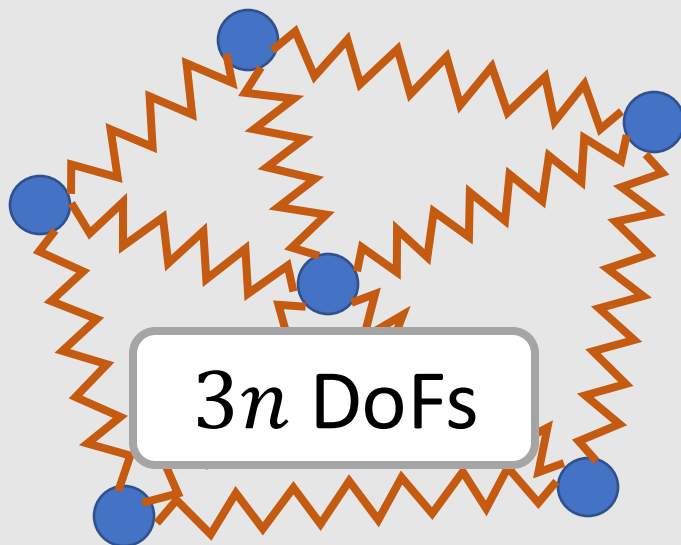


Rigid Body Approximation

- If deformation is negligible, rigid body approximation makes sense

$\vec{x}_{cg}(t)$: the center of gravity's position

$R(t)$: rotation

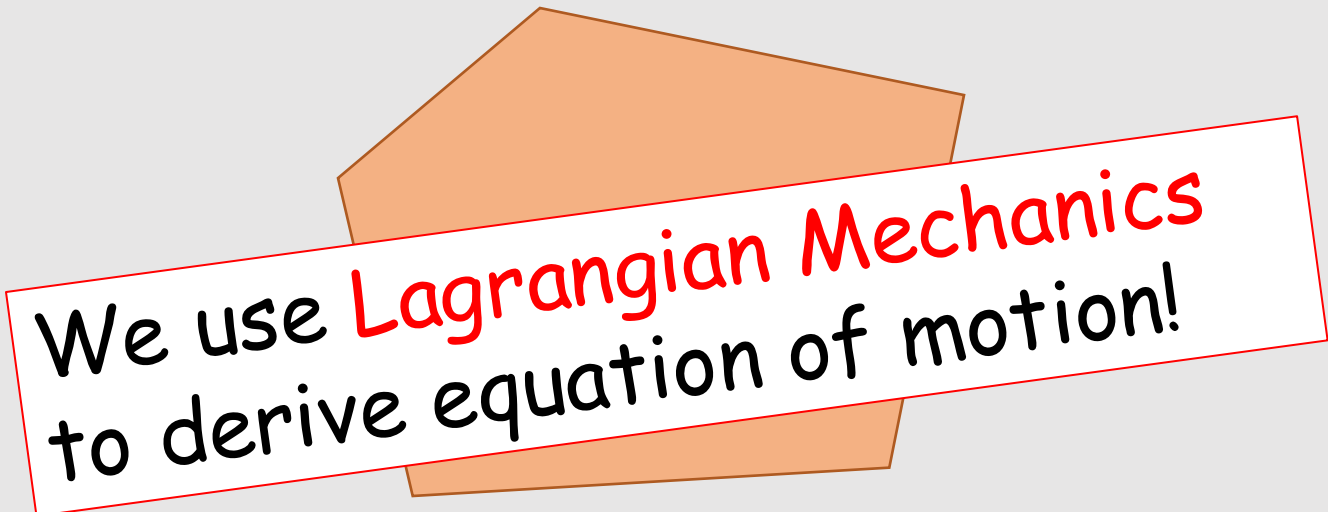


Rigid Body Approximation

- Equation of motion for rigid body?

$p(t)$: the center of gravity's position

$R(t)$: rotation



We use **Lagrangian Mechanics**
to derive equation of motion!

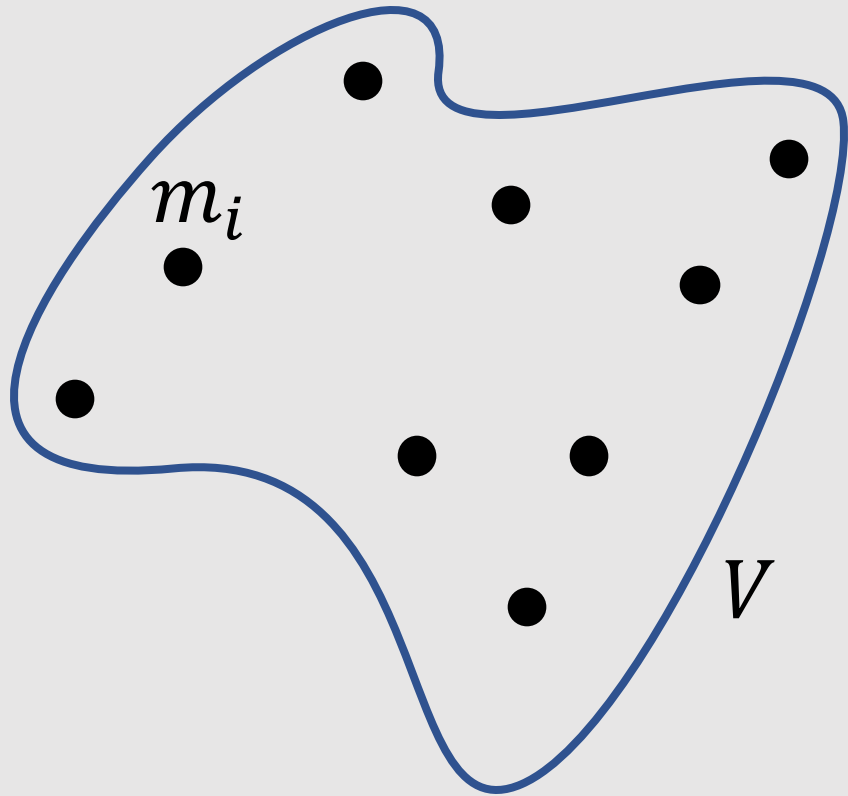
Rigid Body is just an Approximation

- Everything deforms as a reaction to force



Mass (質量)

- Total weight of the object



$$\begin{aligned} M &= m_1 + \cdots + m_i \\ &= \sum_i m_i \end{aligned}$$

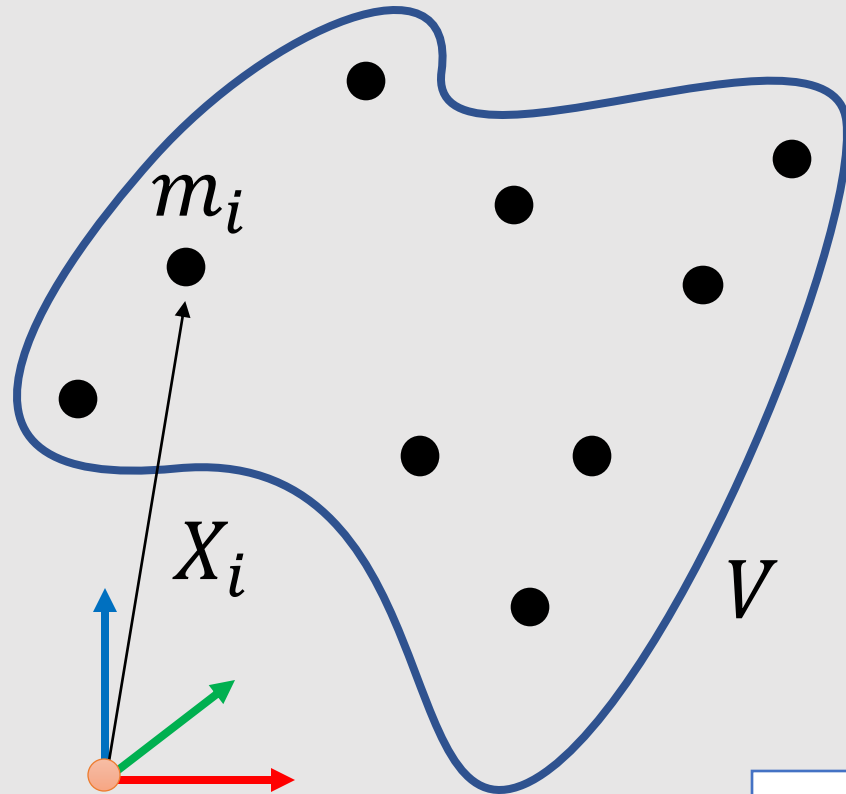
$$M = \int_V \rho \, dV$$



The Center of the Gravity (重心)



- Average of the positions weighted by mass density



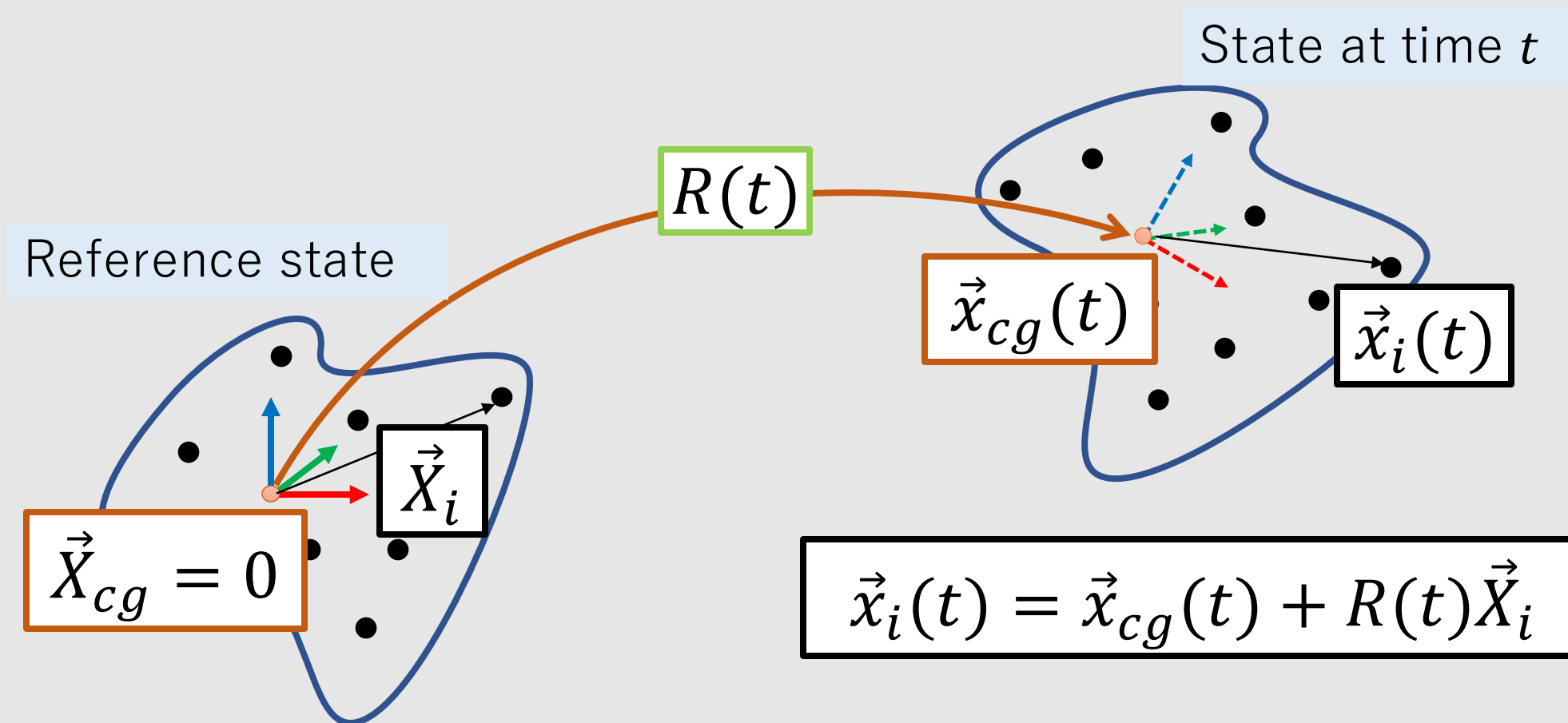
$$\begin{aligned}\vec{X}_{cg} &= \frac{\vec{X}_1 m_1 + \cdots + \vec{X}_i m_i}{m_1 + \cdots + m_i} \\ &= \sum_i \vec{X}_i m_i / \sum_i m_i\end{aligned}$$

$$\vec{X}_{cg} = \int_V \rho \vec{X} dV / \int_V \rho dV$$

$\vec{X}_{cg}$ is the centroid (重心) if ρ is constant

Transformation of a Point on a Rigid Body

- For simplicity, let's put $\vec{X}_{cg}$ at the origin of coordinate: $\vec{X}_{cg} = 0$



Linear Momentum (運動量)

too much
equations....



$$\vec{P} = \sum_i m_i \vec{v}_i$$

$$\vec{x}_i(t) = \vec{x}_{cg}(t) + R(t) \vec{X}_i$$

$$\vec{v}_i(t) = \vec{v}_{cg}(t) + \dot{R}(t) \vec{X}_i$$

$$= R \text{ Skew}(\vec{\Omega})$$

$$\vec{P} = \sum_i m_i \vec{v}_{cg}(t) + \sum_i m_i R \text{ Skew}(\vec{\Omega}) \vec{X}_i$$

$$\vec{P} = M \vec{v}_{cg}$$

$$\begin{aligned} &= R \text{ Skew}(\vec{\Omega}) \sum_i m_i \vec{X}_i \\ &= R \text{ Skew}(\vec{\Omega}) \vec{X}_{cg} \\ &= R \text{ Skew}(\vec{\Omega}) 0 = 0 \end{aligned}$$

I'm full of energy!



Kinetic Energy (運動エネルギー)

$$\mathcal{K} = \frac{1}{2} \sum_i m_i \vec{v}_i^T \vec{v}_i$$

$$\vec{v}_i(t) = \vec{v}_{cg}(t) + \dot{R}(t) \vec{X}_i$$

$$\begin{aligned} &= R \operatorname{Skew}(\vec{\Omega}) \vec{X}_i \\ &= -R \operatorname{Skew}(\vec{X}_i) \vec{\Omega} \end{aligned}$$

$$\mathcal{K} = \frac{1}{2} \sum_i m_i \vec{v}_{cg}^T \vec{v}_{cg} + \frac{1}{2} \sum_i m_i \{R \operatorname{Skew}(\vec{X}_i) \vec{\Omega}\}^T \{R \operatorname{Skew}(\vec{X}_i) \vec{\Omega}\}$$

inertia tensor

$$\mathcal{K} = \frac{1}{2} M \|\vec{v}_{cg}\|^2 + \frac{1}{2} \vec{\Omega}(t)^T \left\{ - \sum_i m_i \operatorname{Skew}(\vec{X}_i) \operatorname{Skew}(\vec{X}_i) \right\} \vec{\Omega}(t)$$

Momentum (運動量)



$$\vec{L} = \sum_i \vec{x}_i \times (m_i \vec{v}_i)$$

$$\vec{x}_i(t) = \vec{x}_{cg}(t) + R(t) \vec{X}_i$$

$$\vec{v}_i(t) = \vec{v}_{cg}(t) + \dot{R}(t) \vec{X}_i$$

$$= R \operatorname{Skew}(\vec{\Omega})$$

$$\vec{L} = \vec{x}_i \times \sum_i m_i \vec{v}_{cg} + \sum_i (R \vec{X}_i) \times (m_i R \operatorname{Skew}(\vec{\Omega}) \vec{X}_i)$$

$$= \operatorname{Skew}(R \vec{X}_i) = R \operatorname{Skew}(\vec{X}_i) R^T$$

$$= -\operatorname{Skew}(\vec{X}_i) \vec{\Omega}$$

inertia tensor

$$\vec{L} = M \vec{x}_{cg}(t) \times \vec{v}_{cg}(t) + R(t) \left\{ - \sum_i m_i \operatorname{Skew}(\vec{X}_i) \operatorname{Skew}(\vec{X}_i) \right\} \vec{\Omega}(t)$$

Inertia Tensor (慣性テンソル)

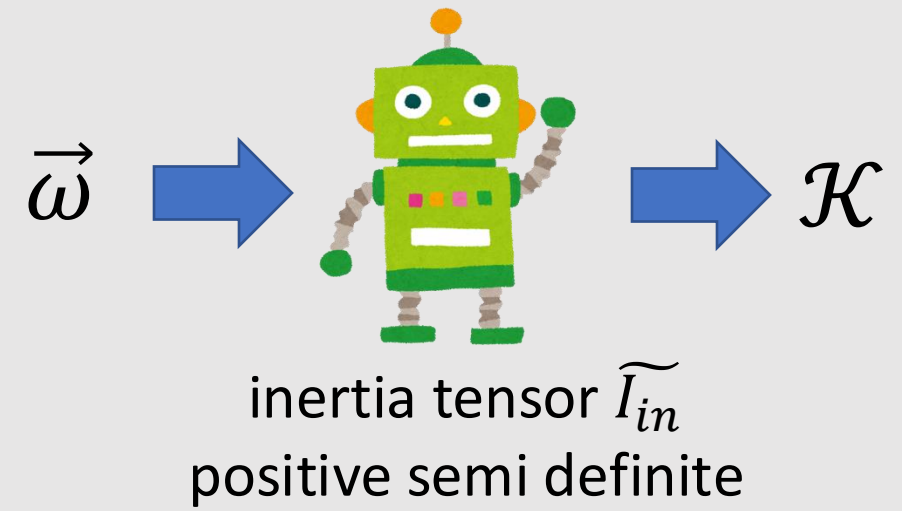
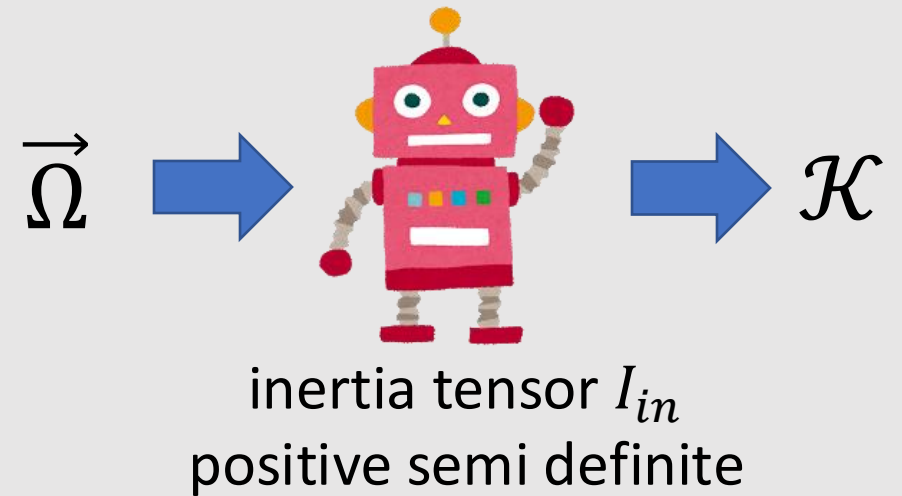
$$I_{in} \equiv -\sum_i m_i \text{Skew}(\vec{X}_i) \text{Skew}(\vec{X}_i)$$

→ $\mathcal{K} = \frac{1}{2} M \|\vec{v}_{cg}\|^2 + \frac{1}{2} \vec{\Omega}^T I_{in} \vec{\Omega}$

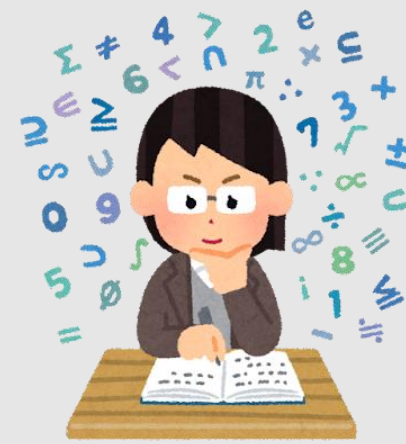
$$\widetilde{I}_{in} \equiv R I_{in} R^T$$

→ $\mathcal{K} = \frac{1}{2} M \|\vec{v}_{cg}\|^2 + \frac{1}{2} \vec{\omega}^T \widetilde{I}_{in} \vec{\omega}$

Quadratic form



Equation for Rigid Body



$$\vec{L} = R(t) I_{in} \vec{\Omega}(t)$$

$$\dot{\vec{L}} = \dot{R}(t) I_{in} \vec{\Omega}(t) + R(t) I_{in} \dot{\vec{\Omega}}(t)$$

$$R(t) \text{Skew}(\vec{\Omega}(t))$$

$$R^T(t) \dot{\vec{L}} = \text{Skew}(\vec{\Omega}(t)) I_{in} \vec{\Omega}(t) + I_{in} \dot{\vec{\Omega}}(t)$$

Torque