Matrix Data Structure

Commutative, Associative & Distributive Laws

Associative law $A \odot (B \odot C) = (A \odot B) \odot C$

Distributive law $A \odot (B + C) = A \odot B + A \odot C$

Commutative law $A \odot B = B \odot A$

Matrices don’t (always) commute

Useful Property of Associative Law

Associative law for matrix: $A(BC) = (AB)C$

$$E\left(D\left(C\left(B(Ax)\right)\right)\right) = \underbrace{(EDCBA)}_{K}x$$

Precompute $K = EDCBA$ to efficiently compute Kx for various x

Quiz: What is an Operator without Associative Law?

$$A \odot (B \odot C) \neq (A \odot B) \odot C$$

Gradient Operator Distributes over Addition

$$W = W_1 + W_2 + \cdots = \sum W_i$$

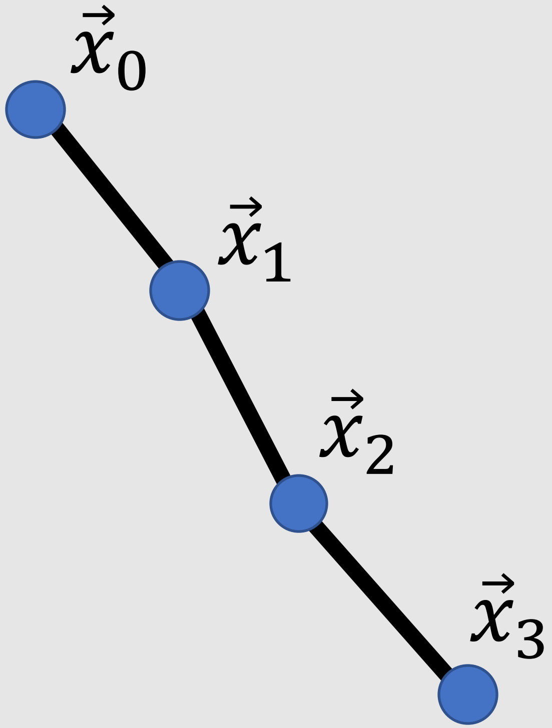
gradient $$\nabla W = \nabla W_1 + \nabla W_2 + \cdots = \sum \nabla W_i$$

hessian $$\nabla^2 W = \nabla^2 W_1 + \nabla^2 W_2 + \cdots = \sum \nabla^2 W_i$$

Sparsity of a Hessian Matrix: Polyline

$$\nabla^2 W(\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x}_3) = \nabla^2 W_1(\vec{x}_0, \vec{x}_1) + \nabla^2 W_2(\vec{x}_1, \vec{x}_2) + \nabla^2 W_3(\vec{x}_2, \vec{x}_3)$$

$$\begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}$$

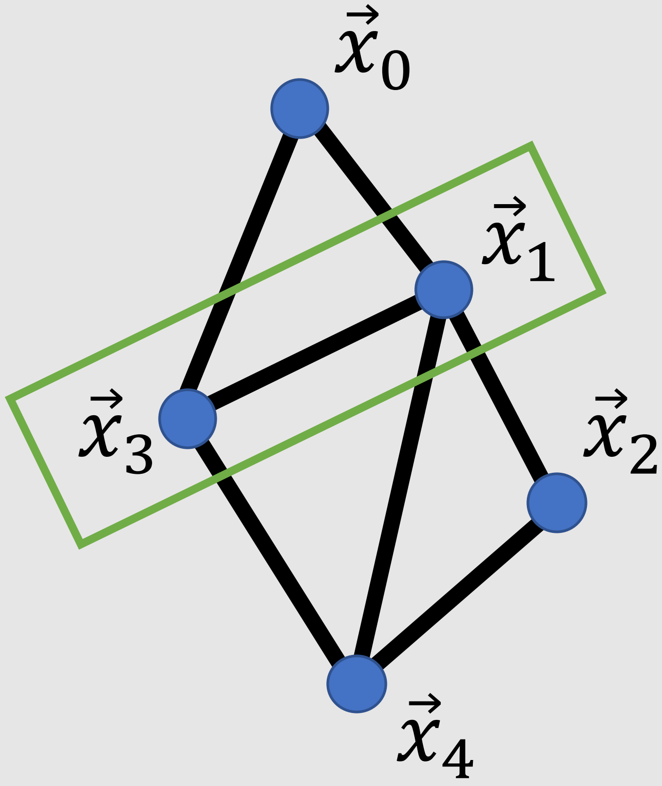


$$\begin{bmatrix} * & * & 0 & 0 \\ * & * & * & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix}$$

band matrix

Sparsity of a Hessian Matrix: Edges

$$\nabla^2 W(\vec{x}_0,\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) = \nabla^2 W'(\vec{x}_0,\vec{x}_1) + \nabla^2 W'(\vec{x}_0,\vec{x}_3) + \nabla^2 W'(\vec{x}_1,\vec{x}_2)$$
$$+\nabla^2 W'(\vec{x}_1,\vec{x}_3) + \nabla^2 W'(\vec{x}_1,\vec{x}_4) + \nabla^2 W'(\vec{x}_2,\vec{x}_4) + \nabla^2 W'(\vec{x}_3,\vec{x}_4)$$

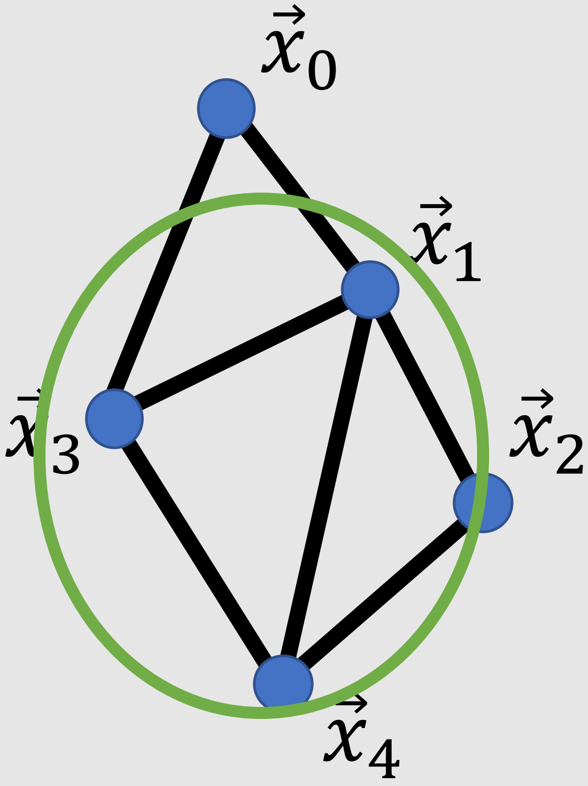


$$\begin{bmatrix} * & * & 0 & * & 0 \\ * & * & * & * & * \\ 0 & * & * & 0 & * \\ * & * & 0 & * & * \\ 0 & * & * & * & * \end{bmatrix}$$

For each row, the non-zero pattern is associated with one-ring neighborhood

Sparsity of a Hessian Matrix: Triangles

$$\nabla^2 W(\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \nabla^2 W'(\vec{x}_0, \vec{x}_3, \vec{x}_1) + \boxed{\nabla^2 W'(\vec{x}_1, \vec{x}_3, \vec{x}_4)} + \nabla^2 W'(\vec{x}_1, \vec{x}_4, \vec{x}_2)$$

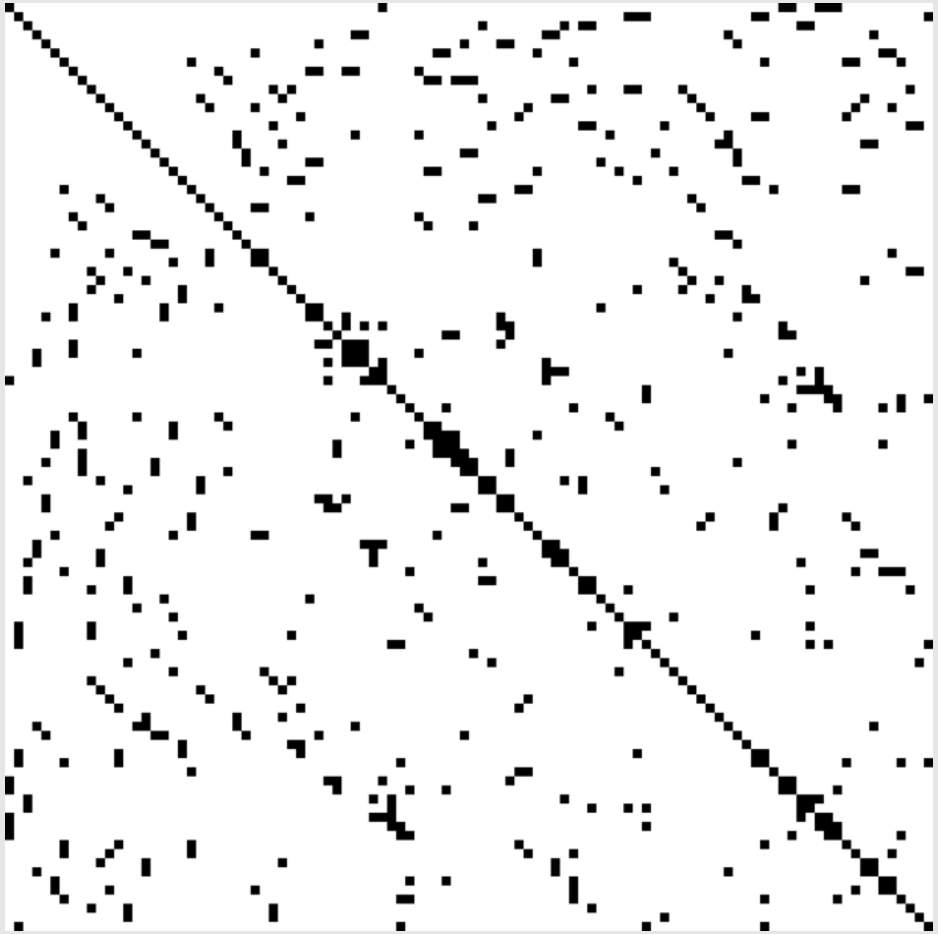


$$\begin{bmatrix} * & * & 0 & * & 0 \\ * & \boxed{*} & * & \boxed{*} & \boxed{*} \\ 0 & * & * & 0 & * \\ * & \boxed{*} & 0 & \boxed{*} & \boxed{*} \\ 0 & \boxed{*} & * & \boxed{*} & \boxed{*} \end{bmatrix}$$

For each row, the non-zero pattern is associated with one-ring neighborhood

Sparse Matrix is Common in Optimization

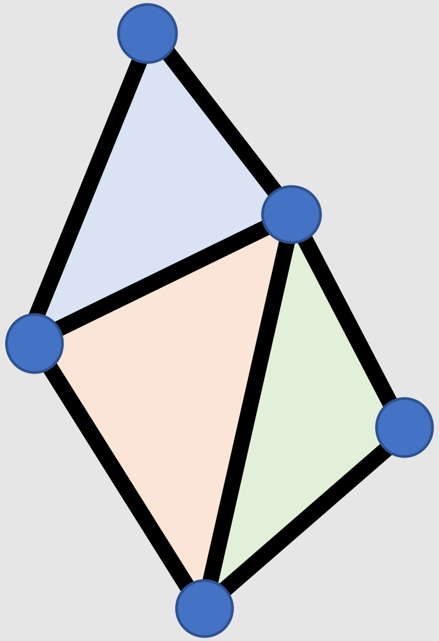
- FEM,FDM produce a very sparse matrix (e.g., 30 entries par row)

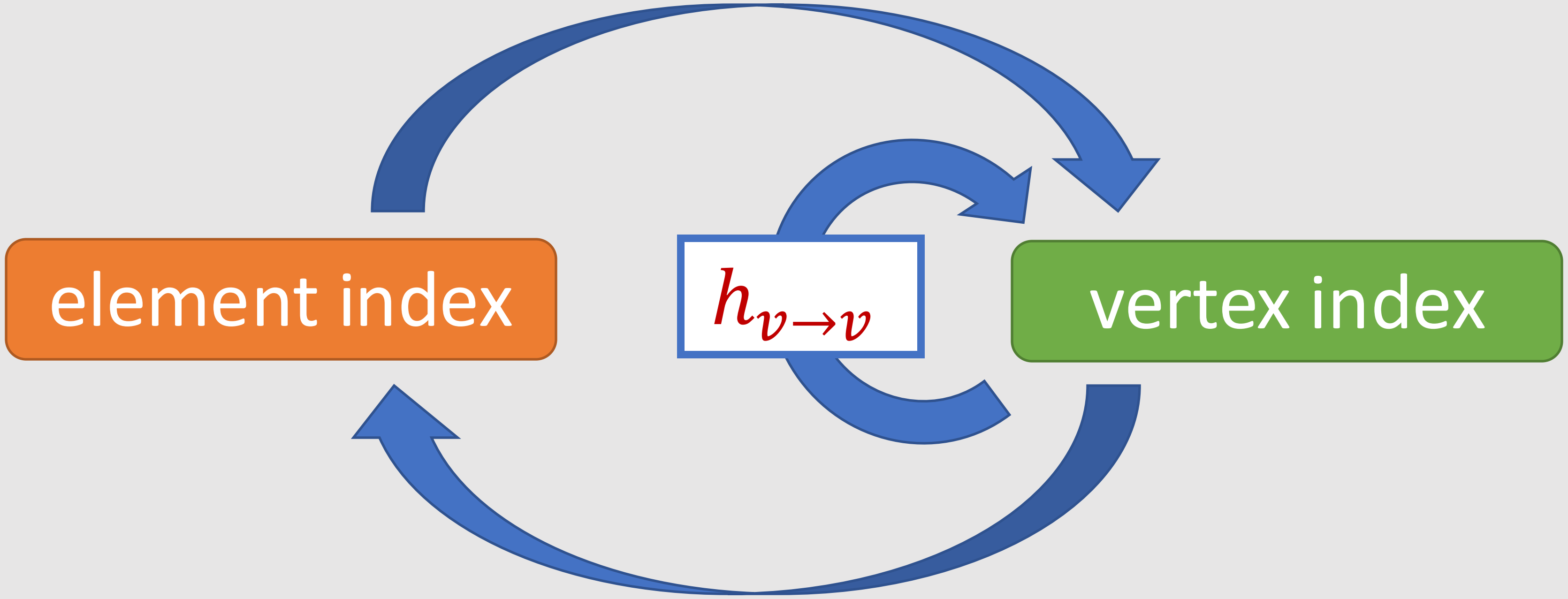


(Wikipedia: Sparse Matrix)

Constructing One-Ring Neighborhood Graph

element to vertex array: $f_{e \to v}$
(regular 2D array)





1. vertex to element array: $g_{v \to e}$
   (jagged array, inverse of $f_{e \to v}$)

2. one-ring neighborhood:
$h_{v \to v} : f_{v \to e}(g_{v \to e}(v))$
(jagged array)

Coordinate (COO) Data Structure

- A.k.a triplet format, 3-tuple format
- Interface for matrix libraries (e.g., Pytorch, Eigen)

$$A=\begin{bmatrix} a & 0 & 0 & b & 0 \\ 0 & c & 0 & 0 & d \\ 0 & 0 & e & 0 & 0 \\ f & 0 & 0 & g & 0 \\ 0 & h & 0 & 0 & i \end{bmatrix}$$

Value	Row	Column
a	0	0
b	0	3
c	1	1
d	1	4
…	…	…

Compressed Sparse Row (CSR) Data Structure

- Data structure for efficient matrix-vector product using jagged array

$$A=\begin{bmatrix} a & 0 & 0 & b & 0 \\ 0 & c & 0 & 0 & d \\ 0 & 0 & e & 0 & 0 \\ f & 0 & 0 & g & 0 \\ 0 & h & 0 & 0 & i \end{bmatrix}$$

A[i][C[B[i]+k]] = D[B[i]+k]

	0	1	2	3	4	5	6	7	8	9
column Index: C =	0	3	1	4	2	0	3	1	4	
value: D =	a	b	c	d	e	f	g	h	i	

indexes of C&D: B =	0	2	4	5	7	9