Geometric Deformation

Orthogonal Procrustes Problem

Optimization of Rotation: $\min_{R} W(R)$

Strategy A: Iteratively update rotation by parameterizing rotational update as $dR = \exp\{\mathrm{Skew}(\vec{\omega})\}$

$$\vec{\omega} = \left[\frac{\partial^2 W}{\partial \vec{\omega}^2}\right]^{-1} \frac{\partial W}{\partial \vec{\omega}}, \qquad R \leftarrow R \exp\{\mathrm{Skew}(\vec{\omega})\}$$

Hard to choose!

Strategy B: Singular Value Decomposition (SVD) can find the optimal rotation if $W(R)$ takes the form of $\|RA - B\|_F$

Optimizing Rotation for Quadratic Metric

$$R_{opt} = \underset{R}{\operatorname{argmin}} \|RA - B\|_F\,, \qquad \text{where } R^T R = I$$

$$= \underset{R}{\operatorname{argmin}} \|RA - B\|_F^2 = \underset{R}{\operatorname{argmin}} \langle RA - B, RA - B \rangle_F$$

$$= \underset{R}{\operatorname{argmin}} \{\|A\|_F^2 - 2\langle RA, B \rangle_F + \|B\|_F^2\}$$

$$= \underset{R}{\operatorname{argmax}} \langle RA, B \rangle_F = \underset{R}{\operatorname{argmax}} \langle R, BA^T \rangle_F$$

Optimizing Rotation for Quadratic Metric

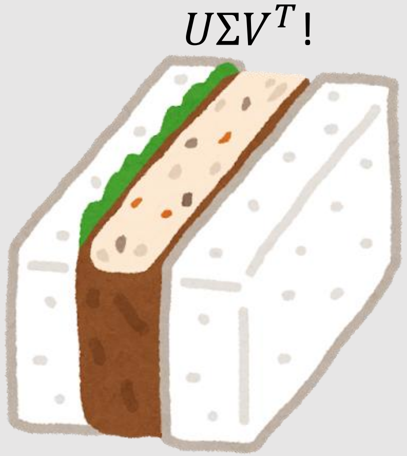
$$
\begin{aligned}
R_{opt} &= \underset{R}{\operatorname{argmax}} \langle R, BA^T \rangle_F \\
&= \underset{R}{\operatorname{argmax}} \langle R, U\Sigma V^T \rangle_F = \underset{R}{\operatorname{argmax}} \langle U^T RV, \Sigma \rangle_F \\
&= \underset{S}{\operatorname{argmax}} \langle S, \Sigma \rangle_F , \qquad \text{where } S^T S = I \\
&= \underset{S}{\operatorname{argmax}} (S_{11}\Sigma_{11} + S_{22}\Sigma_{22} + S_{33}\Sigma_{33}) \\
&= UV^T
\end{aligned}
$$

Solution of Orthogonal Procrustes Problem

Rotation optimization

$$R_{opt} = \underset{R}{\operatorname{argmin}} \|RA - B\|_F \, , \qquad \text{where } R^T R = I$$

$$R_{opt} = UV^T, \qquad \text{where } BA^T = U\Sigma V^T \, (\text{using SVD})$$



As-Rigid-As Possible Deformation

Elastic Energy using Singular Value Decomposition (SVD)

Deformation using SVD

Shape matching deformation

[Müller et al. 2005]

https://www.youtube.com/watch?v=LAoQJ1dhk1w

As-rigid-as possible deformation

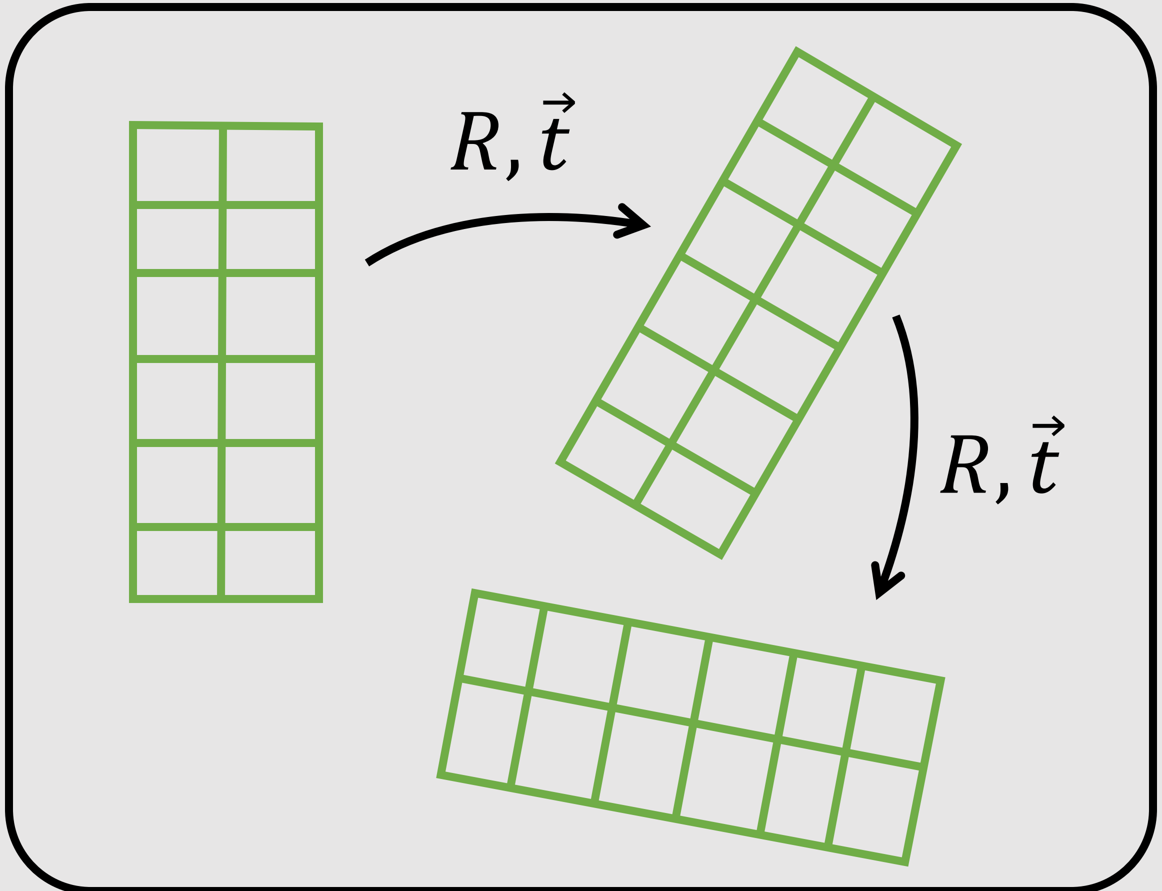
[Sorkine et al. 2007]

https://www.youtube.com/watch?v=ltX-qUjbkdc

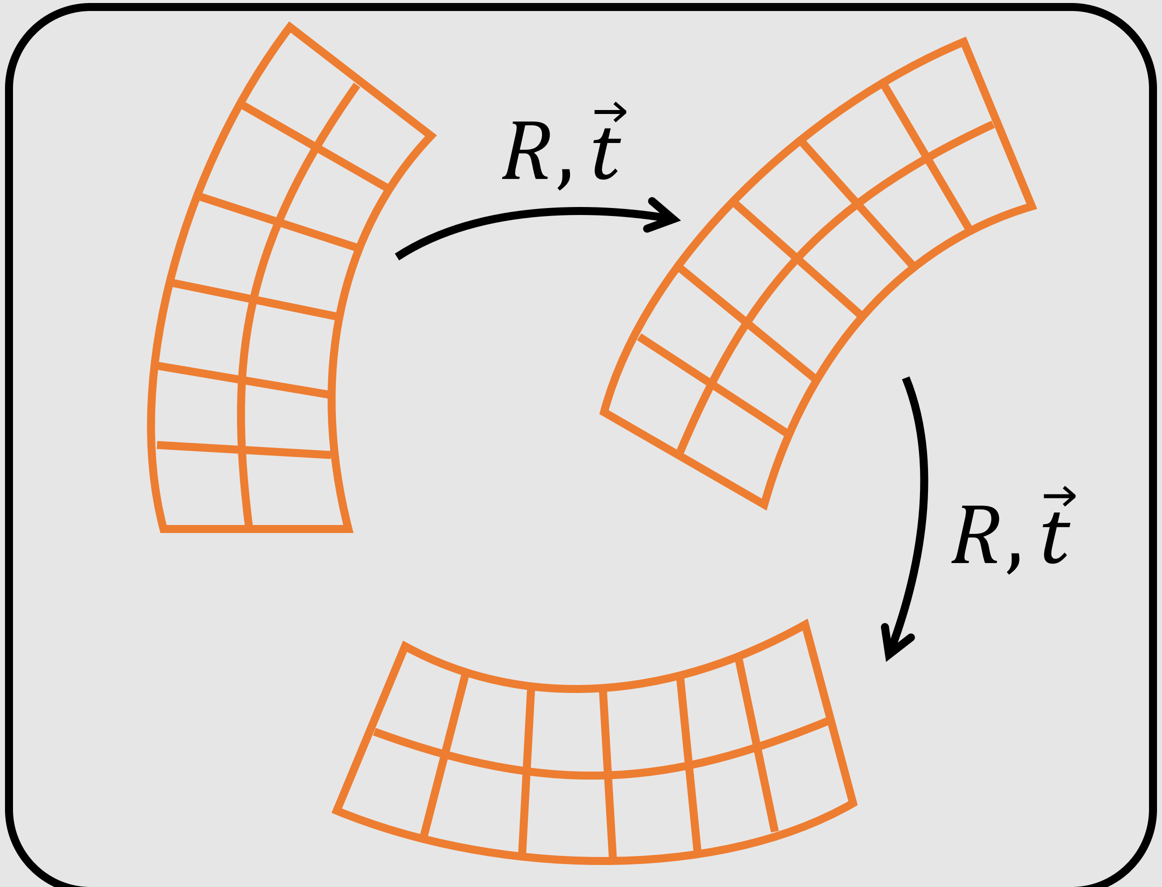
Invariance of the Elastic Energy

- Deformation energy is not affected by rotation R and translation $\vec{t}$

same energy



same energy



How can We Formulate Elastic Energy?

Strategy A: Elastic energy W is a function of eigenvalues of Green-Lagrange strain $E = F^T F - I$

$$W = \|F^T F - I\|_F^2, \qquad \text{where } F = \partial\vec{x}/\partial\vec{X}$$

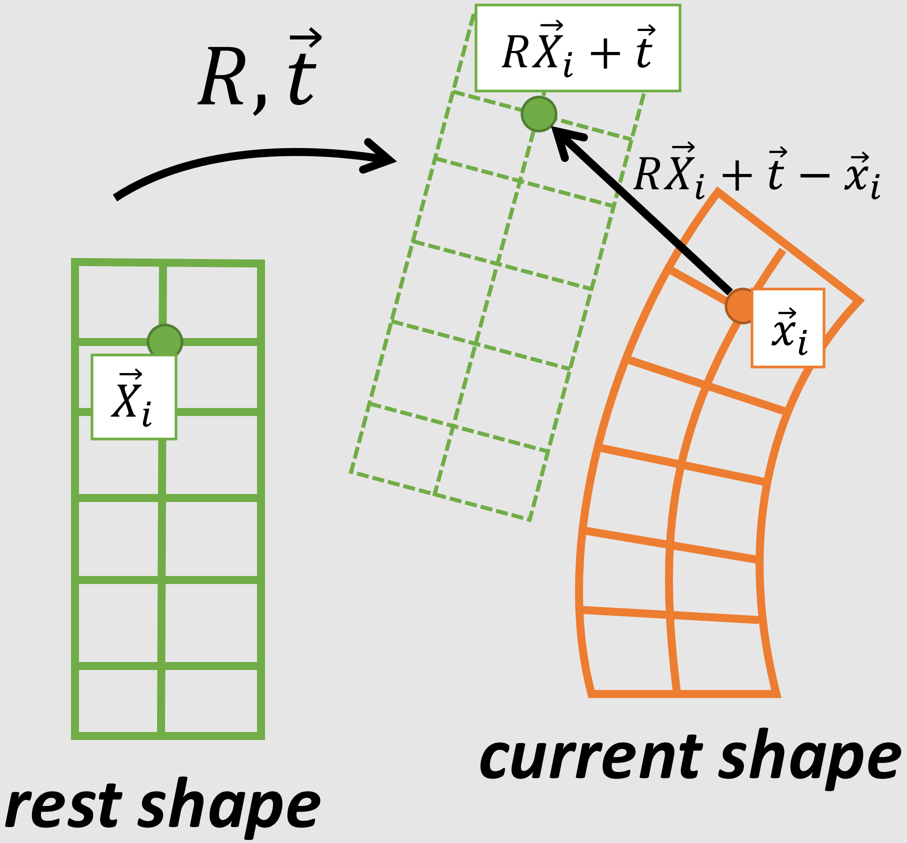
cancel rotation *cancel translation*

Strategy B: Elastic energy W is a sum of square distances after cancelling rotation and translation

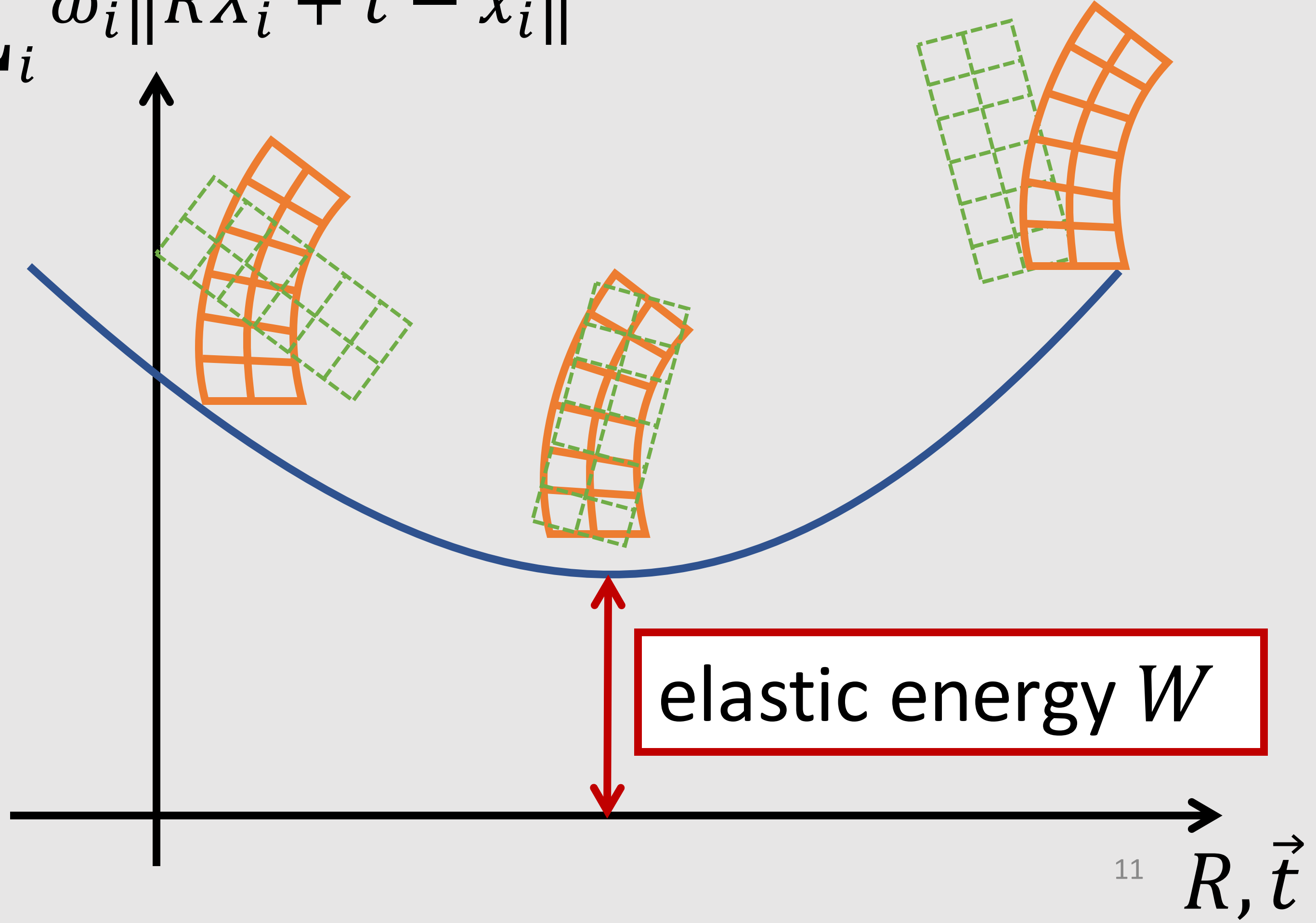
$$W = \min_{R,\vec{t}} \sum_i \omega_i \left\| R\vec{X}_i + \vec{t} - \vec{x}_i \right\|^2$$

Elastic Energy by Sum of Squared Distances

- Given points, if there is a rigid transformation the energy is the same



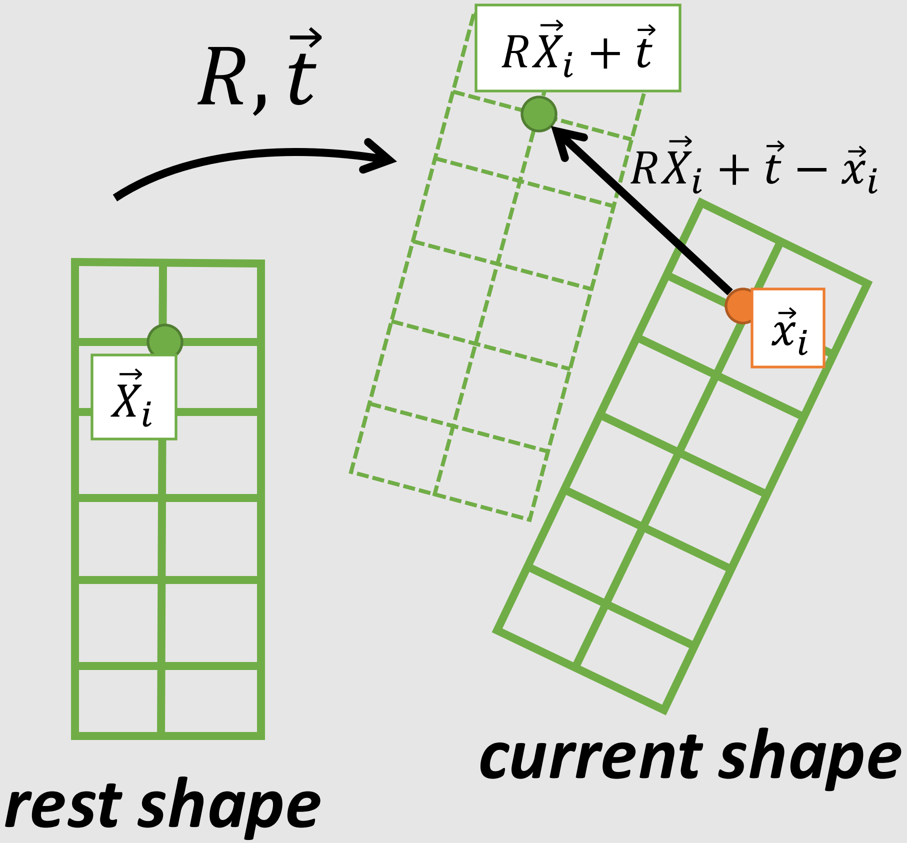
$$\sum_i \omega_i \left\| R\vec{X}_i + \vec{t} - \vec{x}_i \right\|^2$$

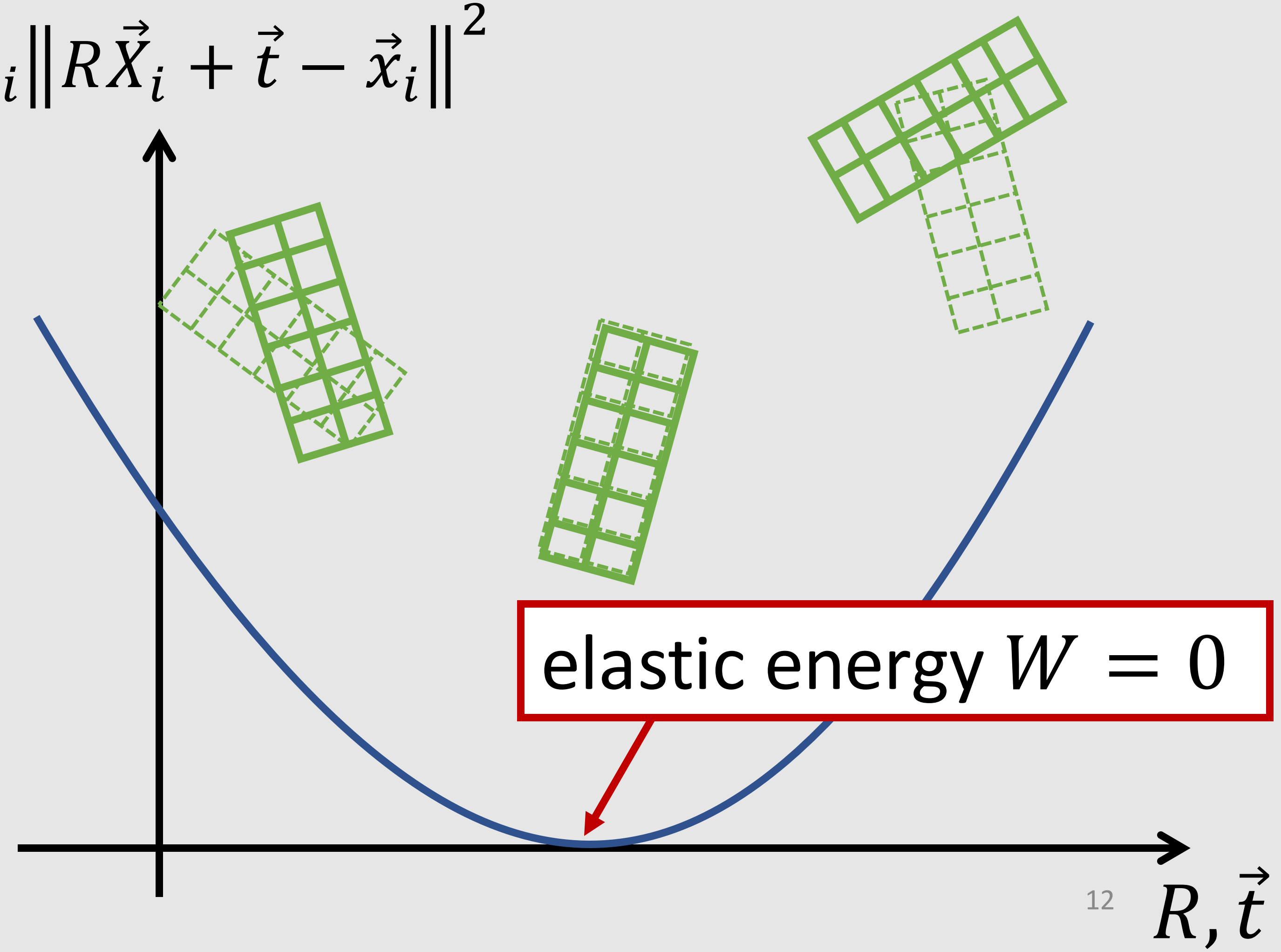


Elastic Energy by Sum of Squared Distances

- Given points, if there is a rigid transformation the energy is the same

$$\sum_i \omega_i \left\| R\vec{X}_i + \vec{t} - \vec{x}_i \right\|^2$$





Compute Elastic Energy W: Optimizing $\vec{t}$

$$\bar{W}(R,\vec{t}) = \sum_i \omega_i \left\| R\vec{X}_i + \vec{t} - \vec{x}_i \right\|^2$$

$$= \sum_i \omega_i \left\| R\vec{X}_i - \vec{x}_i \right\|^2 - \sum_i 2\omega_i \left(\vec{x}_i - R\vec{X}_i\right) \cdot \vec{t} + \sum_i \omega_i \left\| \vec{t} \right\|^2$$

$$\frac{\partial \bar{W}(R,\vec{t})}{\partial \vec{t}} = -\sum_i 2\omega_i \left(\vec{x}_i - R\vec{X}_i\right) + \vec{t} \sum_i 2\omega_i$$

$$\vec{t}_{cg} = \sum_i \omega_i \vec{x}_i \Big/ \sum_i \omega_i$$

$$\vec{T}_{cg} = \sum_i \omega_i \vec{X}_i \Big/ \sum_i \omega_i$$

$$\frac{\partial \bar{W}(R,\vec{t})}{\partial \vec{t}} = 0 \Rightarrow \boxed{\vec{t}_{opt} = \sum_i \omega_i \left(\vec{x}_i - R\vec{X}_i\right) \Big/ \sum_i \omega_i = \vec{t}_{cg} - R\vec{T}_{cg}}$$

Compute Elastic Energy *W*: Optimizing *R*

$$\bar{W}(R, \vec{t}_{opt}) = \sum_i \omega_i \left\| R\vec{X}_i + \vec{t}_{opt} - \vec{x}_i \right\|^2 = \sum_i \omega_i \left\| R(\vec{X}_i - \vec{T}_{cg}) - (\vec{x}_i - \vec{t}_{cg}) \right\|^2$$

$$= \sum_i \left\| R\sqrt{\omega_i}(\vec{X}_i - \vec{T}_{cg}) - \sqrt{\omega_i}(\vec{x}_i - \vec{t}_{cg}) \right\|^2$$

$$= \|RA - B\|_F$$

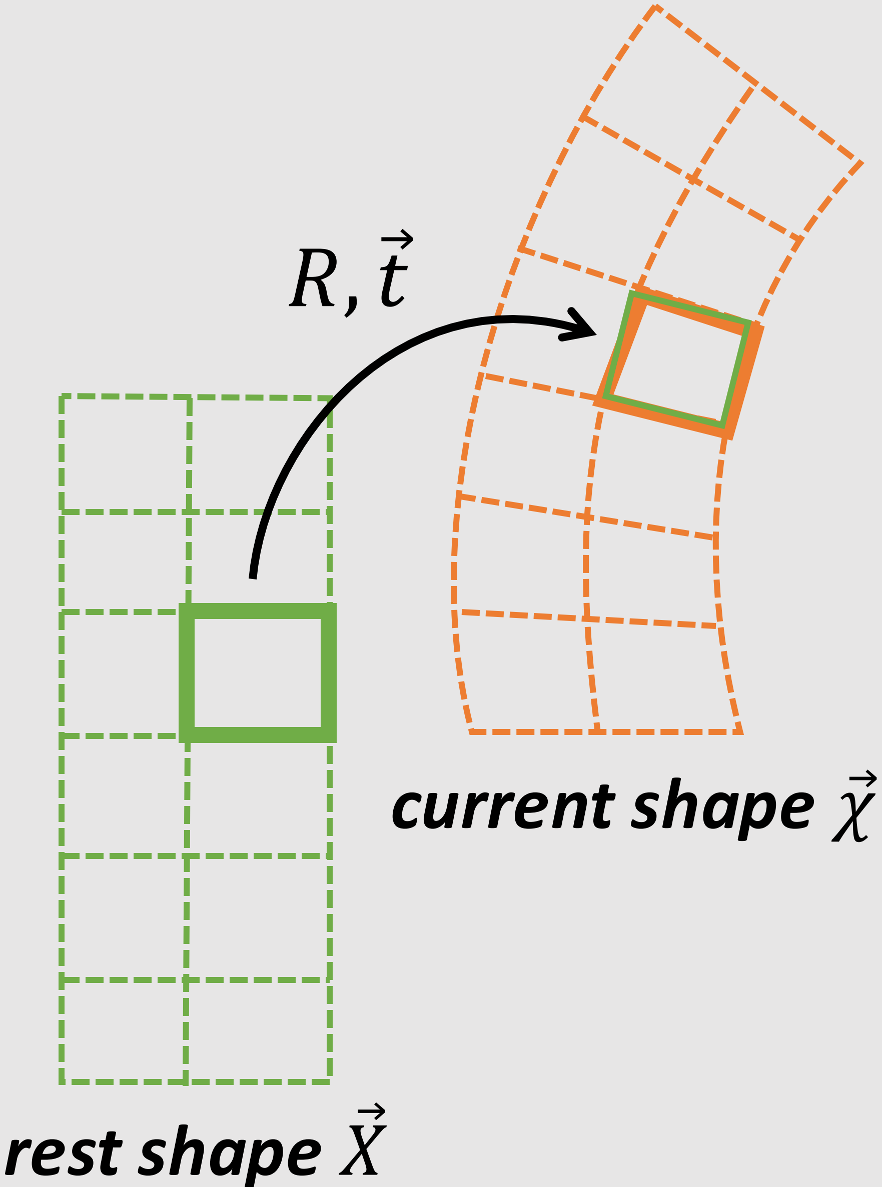
$$A = \left[\sqrt{\omega_1}(\vec{X}_i - \vec{T}_{cg}), \sqrt{\omega_2}(\vec{X}_i - \vec{T}_{cg}), \cdots\right]$$

$$B = \left[\sqrt{\omega_1}(\vec{x}_i - \vec{t}_{cg}), \sqrt{\omega_2}(\vec{x}_i - \vec{t}_{cg}), \cdots\right]$$

$$BA^T = \sum_i w_i \,(\vec{x}_i - \vec{t}_{cg}) \otimes (\vec{X}_i - \vec{T}_{cg})$$

$$R_{opt} = \underset{R}{\operatorname{argmin}}\, \bar{W}(R, \vec{t}_{opt}) = UV^T, \qquad \text{where } BA^T = U\Sigma V^T$$

Shape Matching Deformation [Müller et al. 2005]



1. compute temporary position

$$\vec{\chi}_i = \vec{x}_i + dt \cdot \vec{v}_i$$

2. For each element, update position

$$\color{red}{R_{opt}, \vec{t}_{opt}} = \min_{R,\vec{t}} \sum_{i \in element} m_i \left\| R\vec{X}_i + \vec{t} - \vec{\chi}_i \right\|^2$$

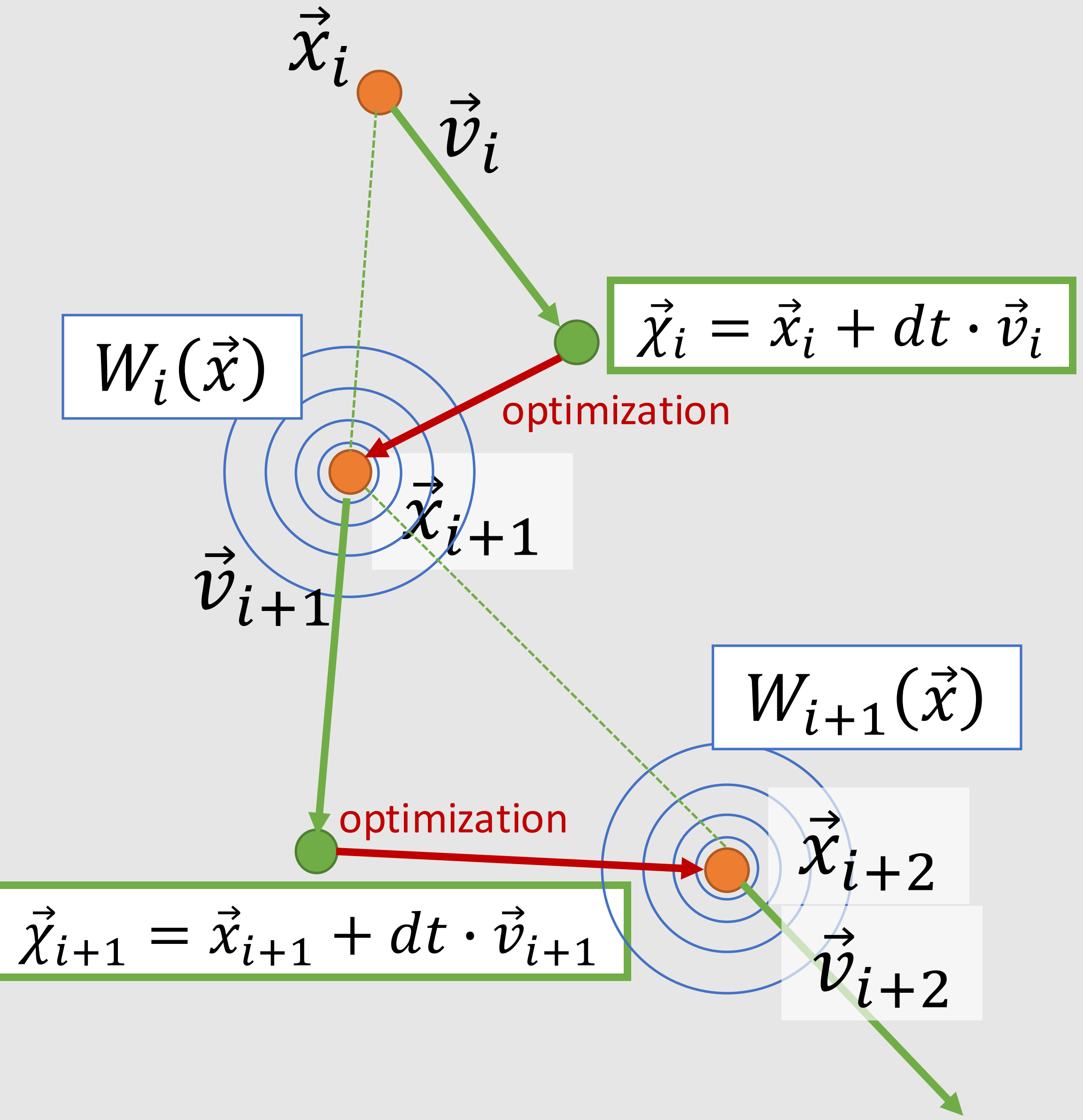
$$\vec{\chi}_i = \color{red}{R_{opt}}\vec{X}_i + \color{red}{\vec{t}_{opt}}$$

3. Set velocity

$$\vec{x}_{i+1} = \vec{\chi}_i, \qquad \vec{v}_{i+1} = \frac{(\vec{x}_{i+1} - \vec{x}_i)}{dt}$$

4. Goto 1

Shape Matching Deformation Artifact



1. compute temporary position

$$\vec{\chi}_i = \vec{x}_i + dt \cdot \vec{v}_i$$

2. optimize $\underline{W_i(\vec{x})}$ to get $\vec{x}_{i+1}$

Inertia term is missing. We should minimize E_i

$$E_i(\vec{x}) = \mathrm{W}(\vec{x}) + \frac{1}{2dt^2}(\vec{x} - \vec{\chi}_i)^T M(\vec{x} - \vec{\chi}_i)$$

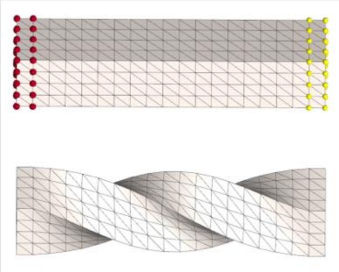
3. Set velocity

$$\vec{v}_{i+1} = \frac{(\vec{x}_{i+1} - \vec{x}_i)}{dt}$$

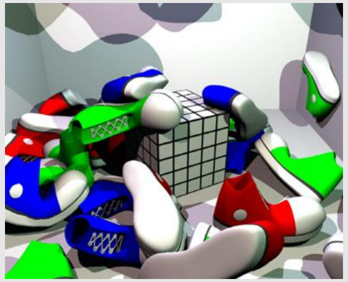
4. Goto 1

Reference

- Olga Sorkine and Marc Alexa. 2007. ***As-rigid-as-possible surface modeling***. In Proceedings of the fifth Eurographics symposium on Geometry processing (SGP '07).



- Matthias Müller, Bruno Heidelberger, Matthias Teschner, and Markus Gross. 2005. ***Meshless deformations based on shape matching***. In ACM SIGGRAPH 2005 Papers (SIGGRAPH '05).



- Wikipedia: “Orthogonal Procrustes problem”, https://en.wikipedia.org/wiki/Orthogonal_Procrustes_problem