Finite Element Method

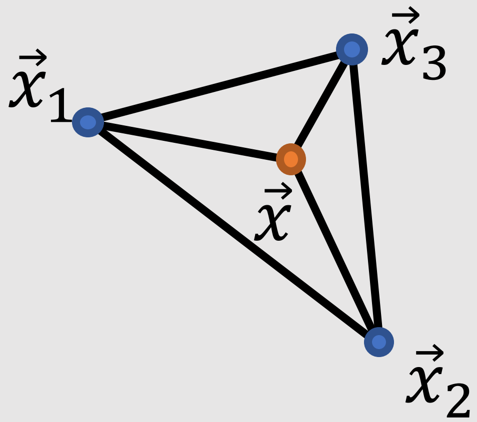
What is Finite Element Method?

- Solution by energy minimization

$$\vec{x}_{solution} = \underset{\vec{x}}{\operatorname{argmin}} W(\vec{x})$$

- Value inside element is interpolated

$$\vec{x} = \sum_{i \in Nodes} w_i \vec{x}_i$$



- Energy is sum of the element-wise energy

$$W(\vec{x}) = \sum_{e \in Elements} W_e(\vec{x})$$

Dirichlet Energy for Triangle Mesh

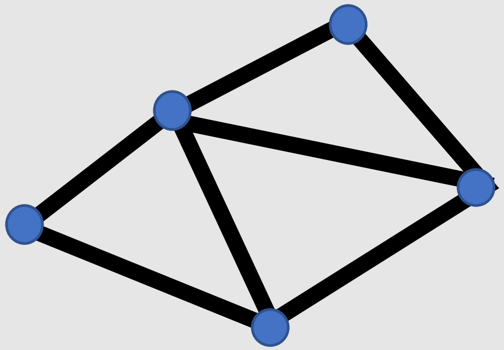
Making the solution as smooth as possible!

Discrete Laplacian

the energy is sum of the squared differences between neighbors

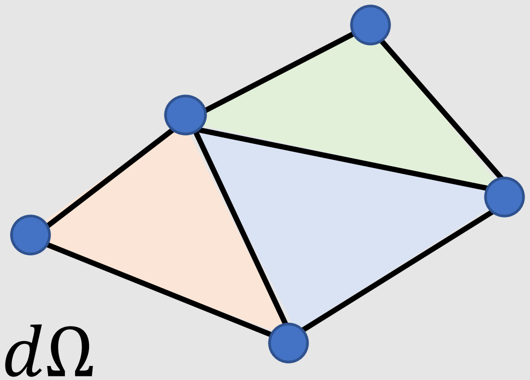
$$W = \frac{1}{2}\sum_{e\subset\mathcal{E}} \left\|x_{e_1} - x_{e_2}\right\|^2$$

$$= \frac{1}{2}\vec{x}^T L \vec{x}$$



Continuous Laplacian

the energy is integration of the squared gradient

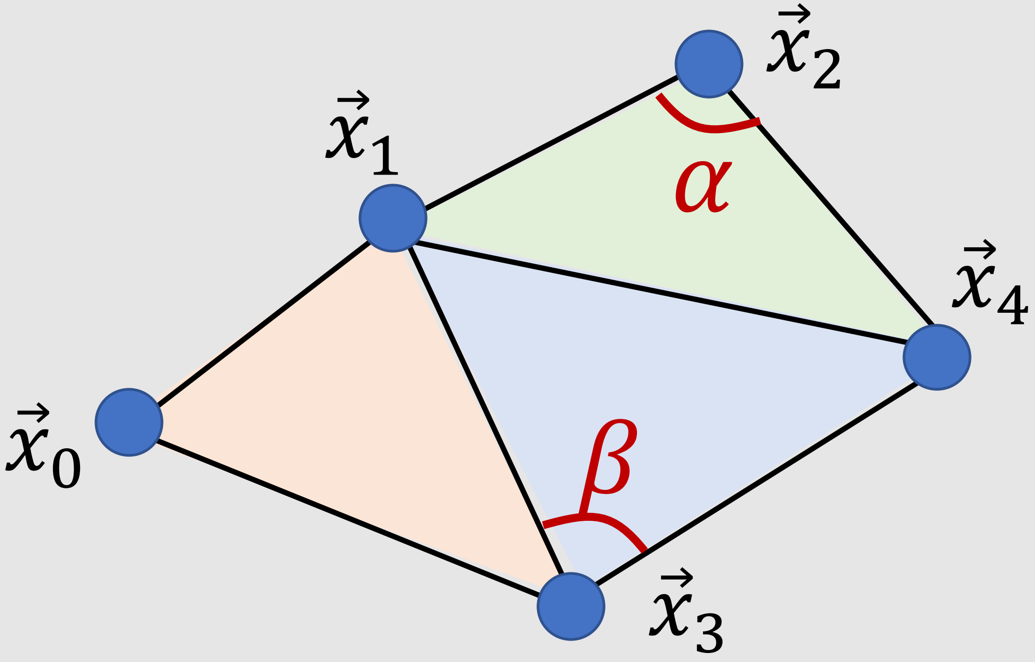


$$W(\phi) = \int_\Omega \nabla\phi \cdot \nabla\phi \, d\Omega$$

$$W_e(\phi) = \int_{\vec{x}\in Tri} \frac{\partial L_a \phi_a}{\partial \vec{x}} \cdot \frac{\partial L_b \phi_b}{\partial \vec{x}} \, d\vec{x}$$

$$= \phi_a \phi_b \int_{\vec{x}\in Tri} \frac{\partial L_a}{\partial \vec{x}} \cdot \frac{\partial L_b}{\partial \vec{x}} \, d\vec{x}$$

Cotangent Weight in Continuous Laplacian



- Continuous Setting

$$W = \frac{1}{2}\sum_{t\subset\mathcal{T}}\int_{\Omega_t} \|\nabla x\|^2 \, d\Omega$$

$$= \frac{1}{2}\vec{x}^T \bar{L}\vec{x}$$

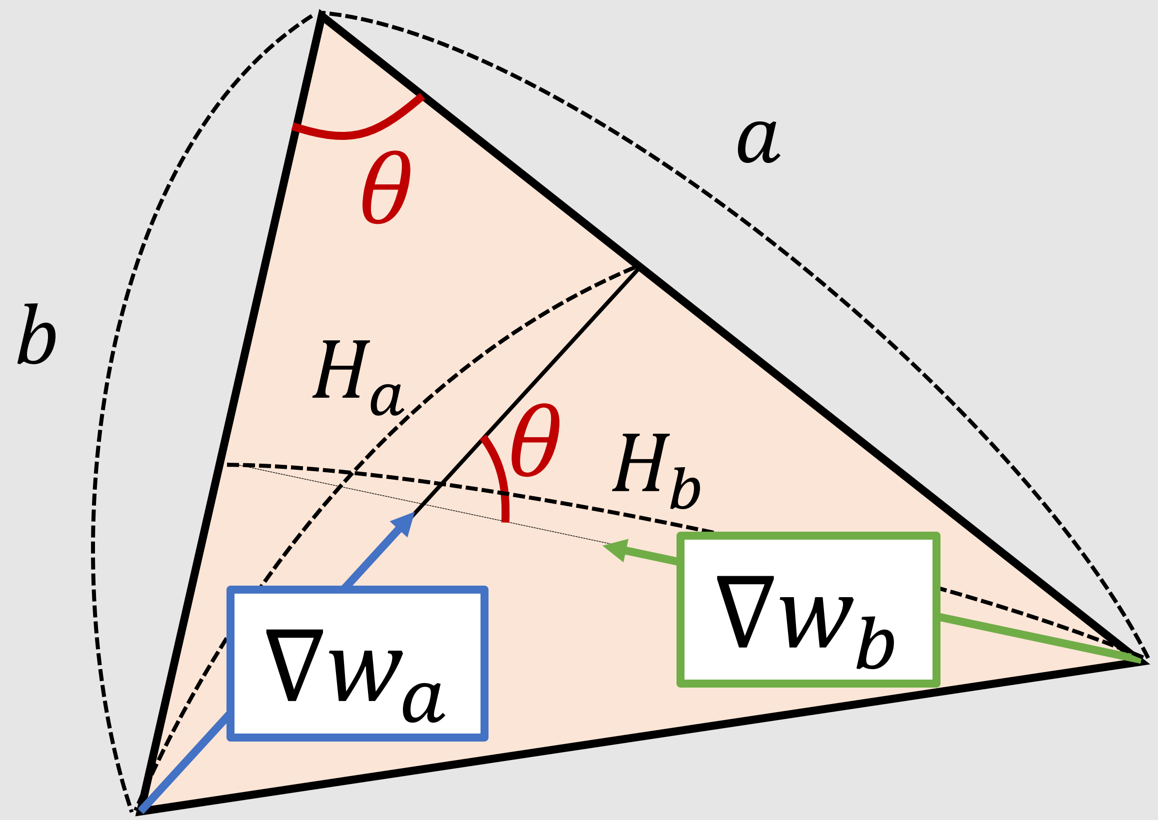
$$\begin{bmatrix} l_{00} & l_{01} & 0 & l_{03} & 0 \\ l_{10} & l_{11} & l_{12} & l_{13} & l_{14} \\ 0 & l_{21} & l_{22} & 0 & l_{24} \\ l_{30} & l_{31} & l_{32} & l_{33} & l_{34} \\ 0 & l_{41} & l_{42} & l_{43} & 0 \end{bmatrix}$$

$$l_{14} = -\frac{1}{2}(\cot\alpha + \cot\beta)$$

$$l_{11} = -(l_{10} + l_{12} + l_{13} + l_{14})$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Why Cotangent?



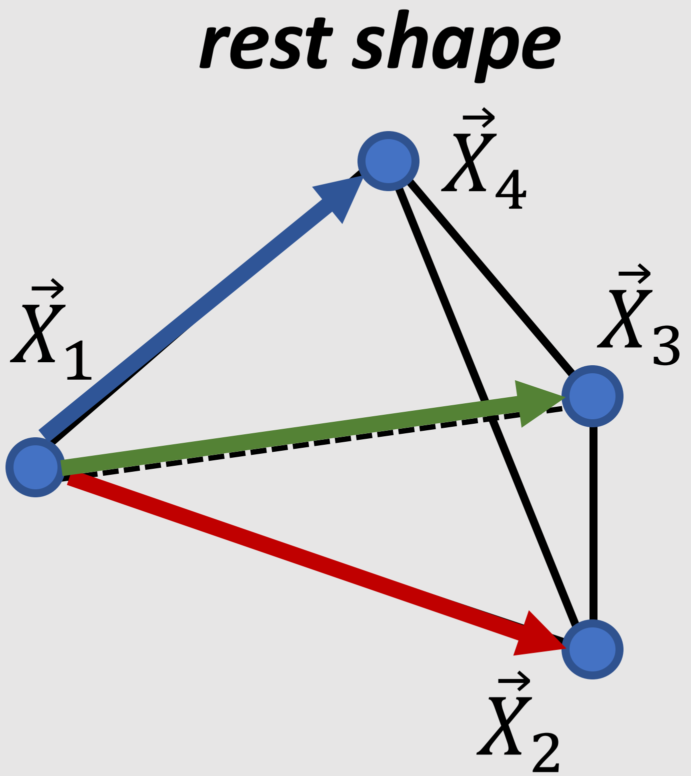
$$Area = \frac{1}{2} ab \sin\theta$$

$$\|\nabla w_a\| = H_a = \frac{a}{2Area}$$

$$\|\nabla w_b\| = H_b = \frac{b}{2Area}$$

$$l_{ij} = \int_{\Omega_t} \nabla w_a \cdot \nabla w_b \, d\Omega = Area \|\nabla w_a\| \|\nabla w_b\| \cos\theta = \frac{\cos\theta}{2\sin\theta} = \frac{\cot\theta}{2}$$

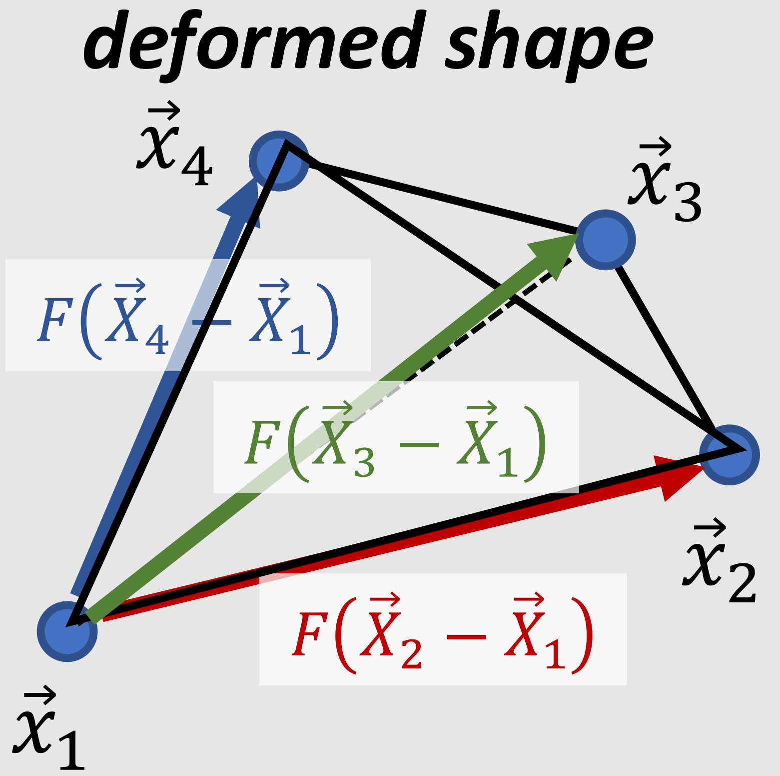
Deformation Gradient Tensor F for Tet.



$$(\vec{x}_2 - \vec{x}_1) = F(\vec{X}_2 - \vec{X}_1)$$

$$(\vec{x}_3 - \vec{x}_1) = F(\vec{X}_3 - \vec{X}_1)$$

$$(\vec{x}_4 - \vec{x}_1) = F(\vec{X}_4 - \vec{X}_1)$$

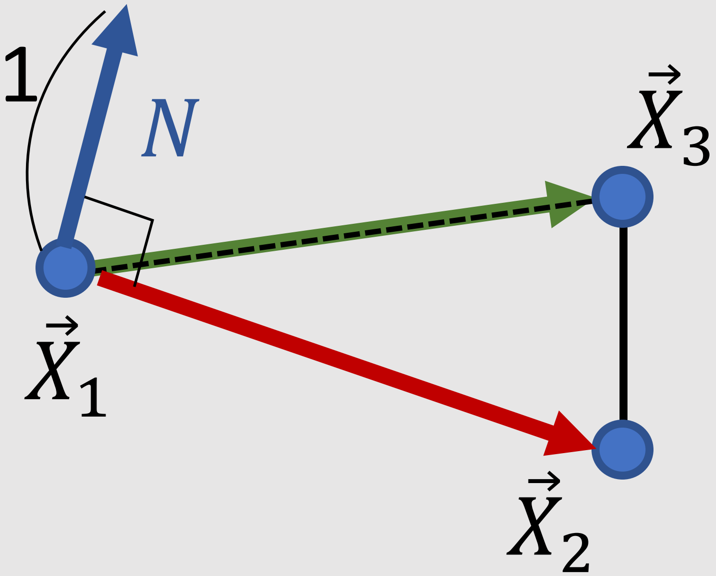


$$[\vec{x}_2 - \vec{x}_1, \vec{x}_3 - \vec{x}_1, \vec{x}_4 - \vec{x}_1] = F[\vec{X}_2 - \vec{X}_1, \vec{X}_3 - \vec{X}_1, \vec{X}_4 - \vec{X}_1]$$

$$F = [\vec{x}_2 - \vec{x}_1, \vec{x}_3 - \vec{x}_1, \vec{x}_4 - \vec{x}_1][\vec{X}_2 - \vec{X}_1, \vec{X}_3 - \vec{X}_1, \vec{X}_4 - \vec{X}_1]^{-1}$$

Deformation Gradient Tensor F for 3D Tri.

rest shape

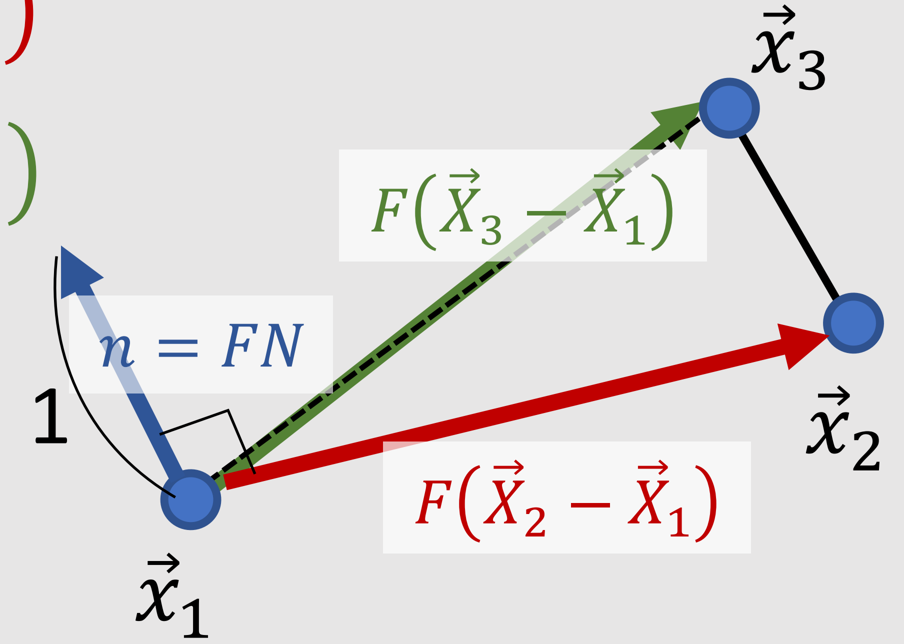


$$(\vec{x}_2 - \vec{x}_1) = F\left(\vec{X}_2 - \vec{X}_1\right)$$

$$(\vec{x}_3 - \vec{x}_1) = F\left(\vec{X}_3 - \vec{X}_1\right)$$

$$n = FN$$

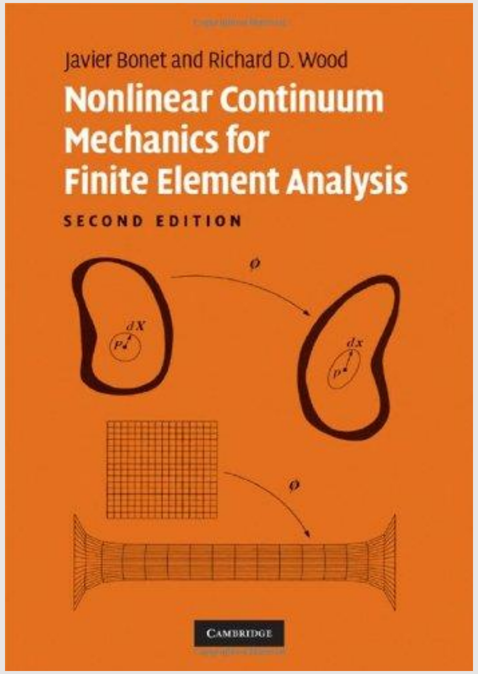
deformed shape



$$[\vec{x}_2 - \vec{x}_1, \vec{x}_3 - \vec{x}_1, n] = F\left[\vec{X}_2 - \vec{X}_1, \vec{X}_3 - \vec{X}_1, N\right]$$

$$F = [\vec{x}_2 - \vec{x}_1, \vec{x}_3 - \vec{x}_1, n]\left[\vec{X}_2 - \vec{X}_1, \vec{X}_3 - \vec{X}_1, N\right]^{-1}$$

Reference



- Bonet, Javier, and Richard D. Wood. 1997. *Nonlinear continuum mechanics for finite element analysis*. Cambridge: Cambridge University Press.