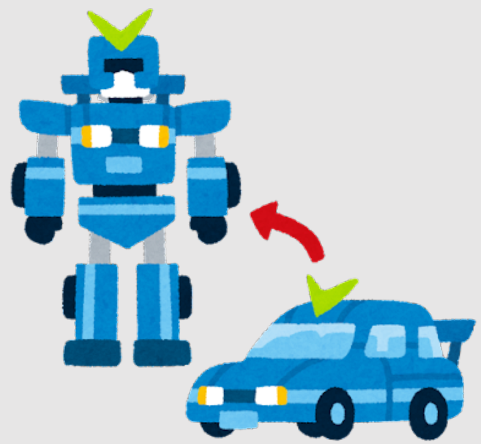
Coordinate Transformation

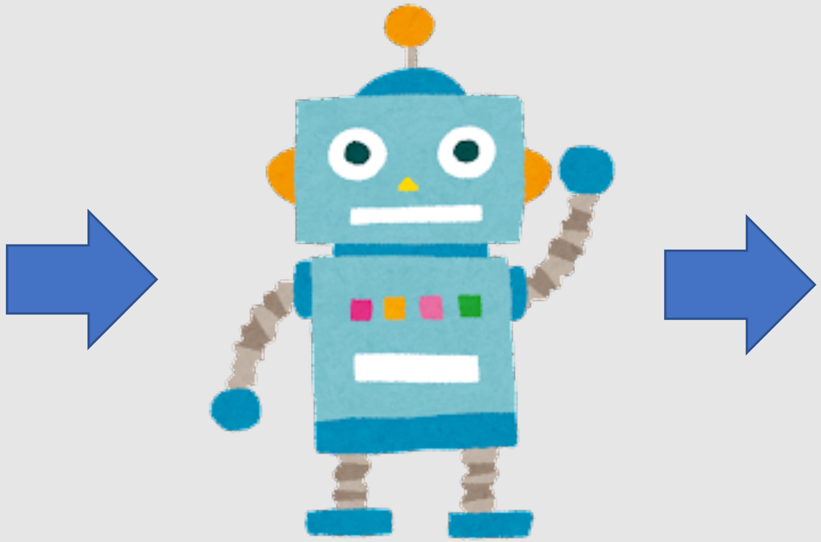


What is Coordinate Transformation?

- Function to input & output coordinates

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

input



$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

output

Basic Coordinate Transformations

- Linear transformation
- Affine transformation
- Homographic transformation

Homogeneous coordinates

Linear Transformation

Definition of linear map

$$f(\vec{p}+\vec{q}) = f(\vec{p}) + f(\vec{q})$$

$$f(\alpha\vec{p}) = \alpha f(\vec{p})$$

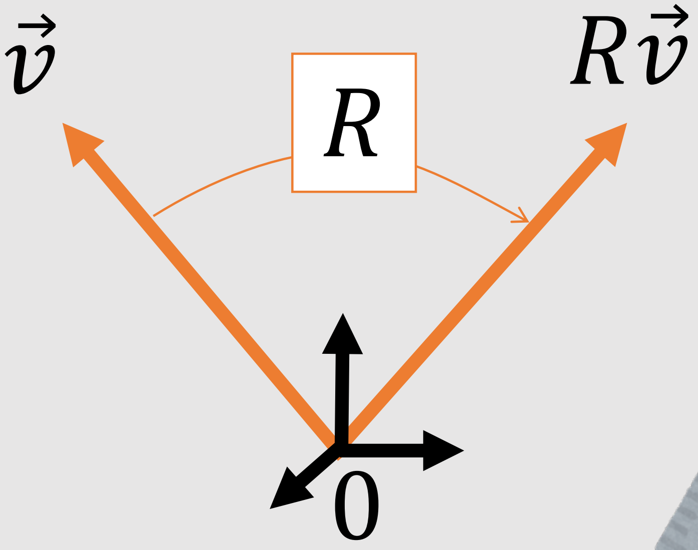
Can be written as
matrix-vector product

$$f(\vec{\mathrm{p}}) = A\vec{p}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Rotation: Preserve Length

- The transformation matrix should be unitary $R^{-1} = R^{T}$



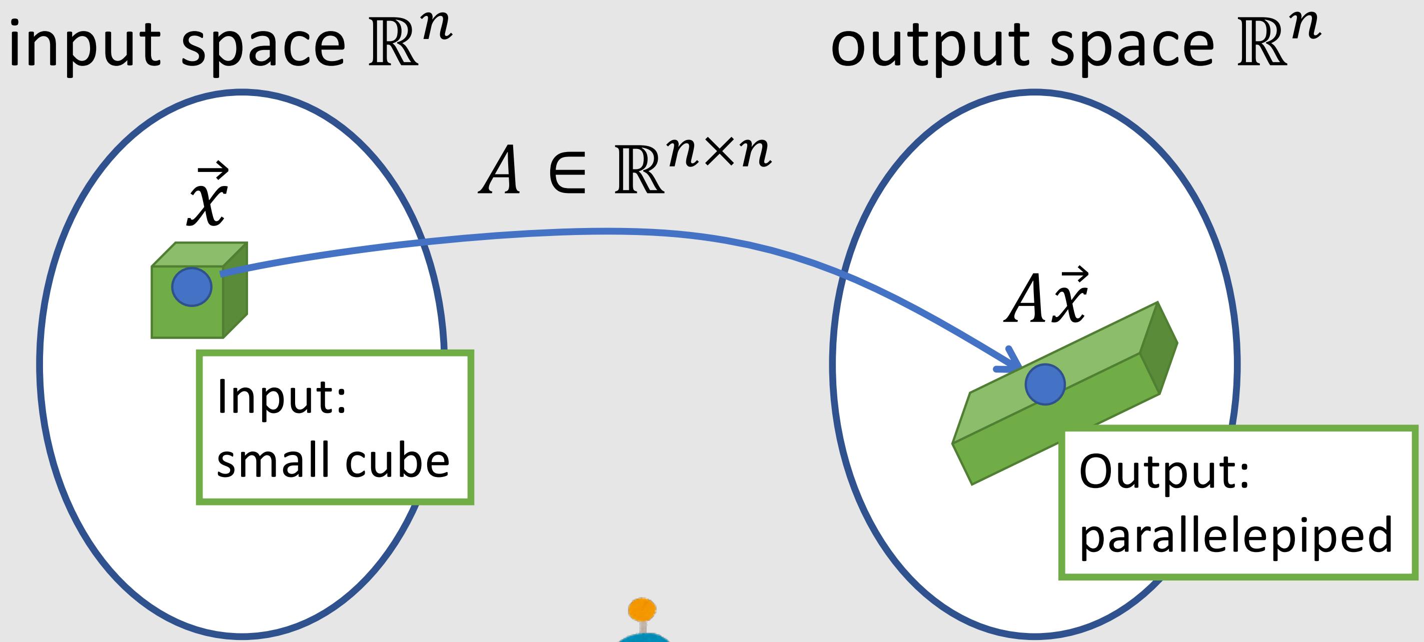
$$\|\vec{v}\|^2 = \|R\vec{v}\|^2$$

$$\vec{v}^T\vec{v} = \vec{v}^T R^T R\vec{v}$$

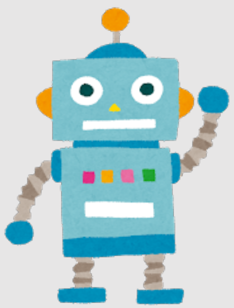
$\vec{v}$ can be arbitrary!

$$R^T R = I$$

Determinant: Volume Change Ratio



Input
volume: dv



Output
volume = $\det(A)\, dv$

Rotation: Preserve Volume

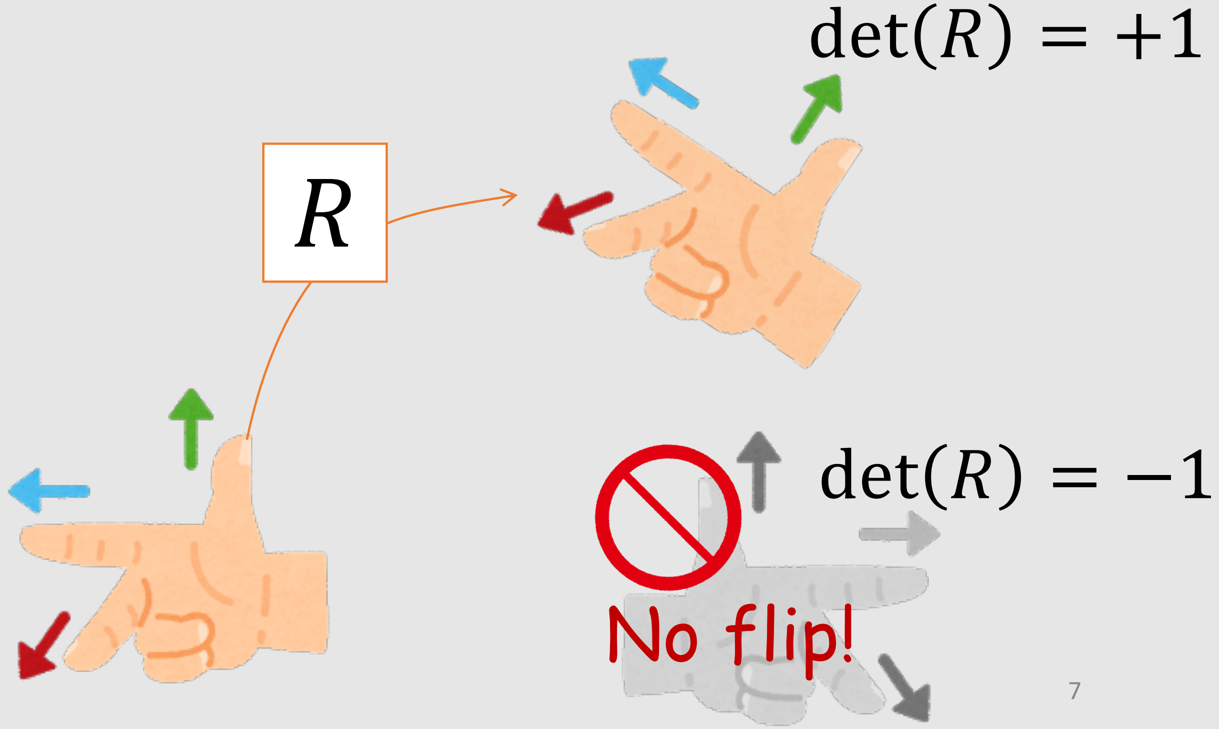
- Rigid transformation R is the one with $\det(R) = +1$

$$R^T R = I$$

$$\det(R^T R) = \det(\mathrm{I})$$

$$\det(R)^2 = 1$$

$$\det(R) = \pm 1$$



Rotation

R_x:

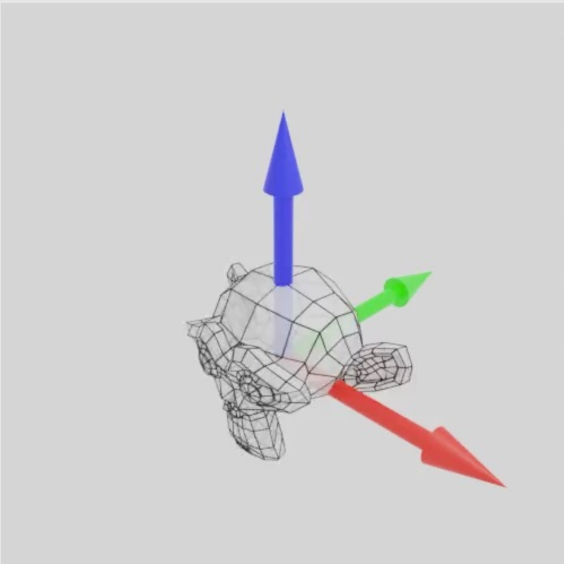
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

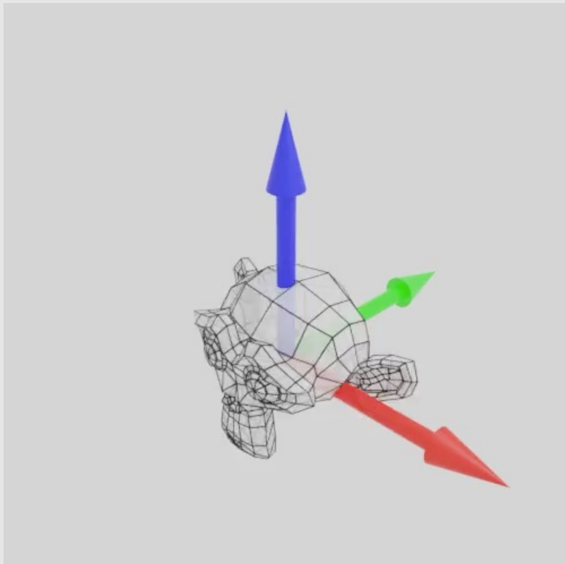
R_y:

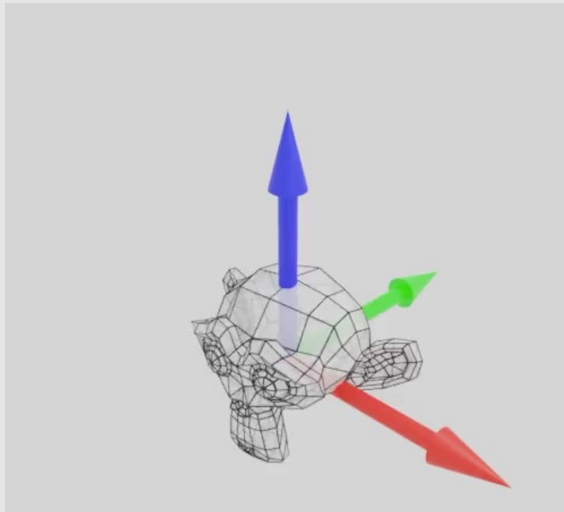
$$\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

R_z:

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



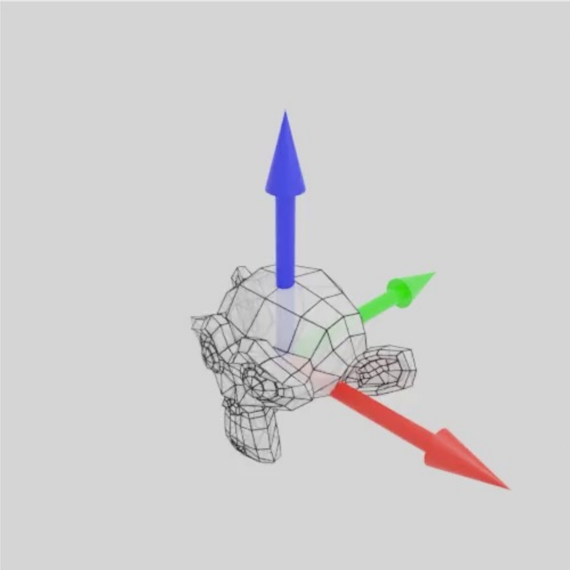




Scaling

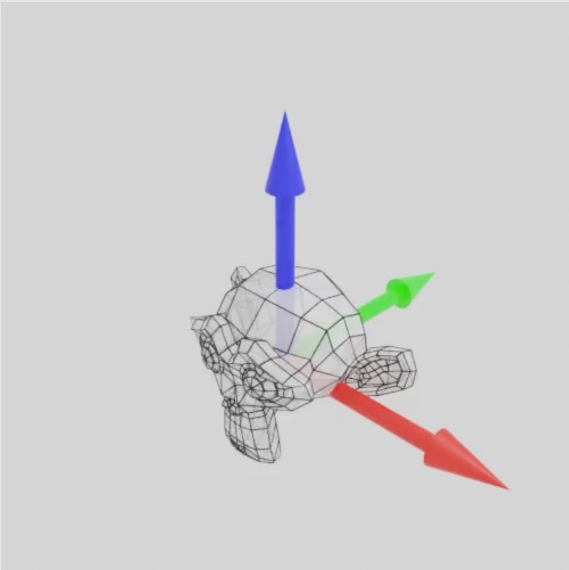
S_x:

$$\begin{pmatrix} s_x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



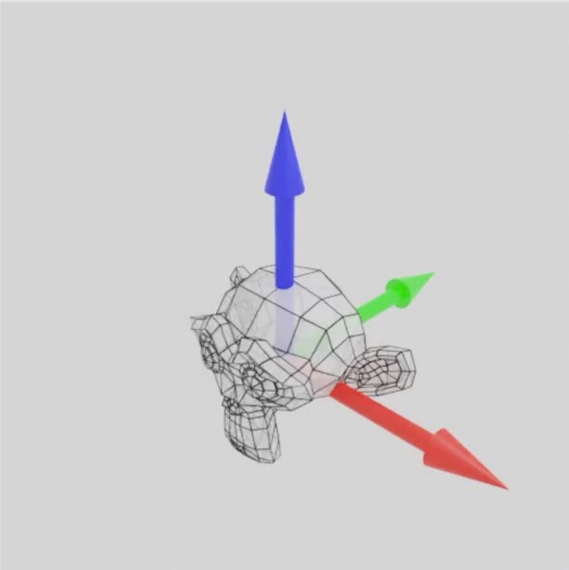
S_y:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



S_z:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s_z \end{pmatrix}$$



Singular Value Decomposition (SVD)

- Any linear transformation is a combination of rotation and scaling

Rotation (U) — Rotation (V^T)

$$A = U\Sigma V^T$$

Scaling (Σ)

Intuition of SVD

- Shear deformation = Rotation + Scaling

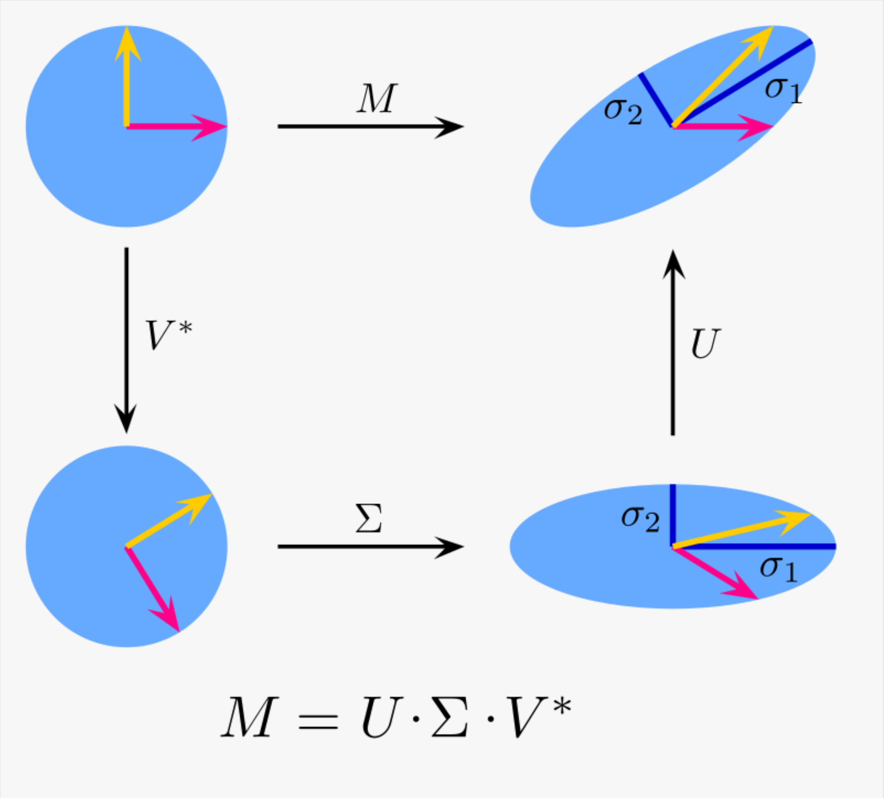


Image Credit : Georg-Johann @ Wikipedia

Affine Transformation

Linear transformation & translation

$$\vec{x}' = K\vec{x} + \vec{t}$$

Affine transformation

$$\begin{pmatrix} \vec{x}' \\ 1 \end{pmatrix} = \begin{bmatrix} K & \vec{t} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \vec{x} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{bmatrix} K_{xx} & K_{xy} & K_{xz} \\ K_{yx} & K_{yy} & K_{yz} \\ K_{yz} & K_{zy} & K_{zz} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{bmatrix} K_{xx} & K_{xy} & K_{xz} & t_x \\ K_{yx} & K_{yy} & K_{yz} & t_y \\ K_{zx} & K_{zy} & K_{zz} & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Properties of Affine Transformation

- Composite of two affine transformations makes an affine transformation

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Associative property: $(A_1 A_2) A_3 = A_1 (A_2 A_3)$

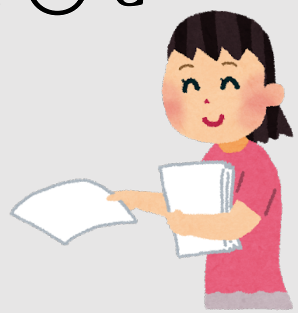
- Inverse is also an affine transformation

$$\begin{bmatrix} K^{-1} & -K^{-1}\vec{t} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} K & \vec{t} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}$$

Commutative, Associative & Distributive Laws

Associative law $A \odot (B \odot C) = (A \odot B) \odot C$

Distributive law $A \odot (B + C) = A \odot B + A \odot C$



Commutative law $A \odot B = B \odot A$

Matrices don’t (always) commute

Useful Property of Associative Law

Associative law for matrix: $A(BC) = (AB)C$

$$E\left(D\left(C\left(B(Ax)\right)\right)\right) = \underbrace{(EDCBA)}_{K}x$$

Precompute $K = EDCBA$ to efficiently compute Kx for various x

Affine Transformation in 2D

- General form of 2D affine transformation

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

scale

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

rotation

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

translation

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

Composite of Affine Transformations in 2D

translation (a, b) and rotation with θ

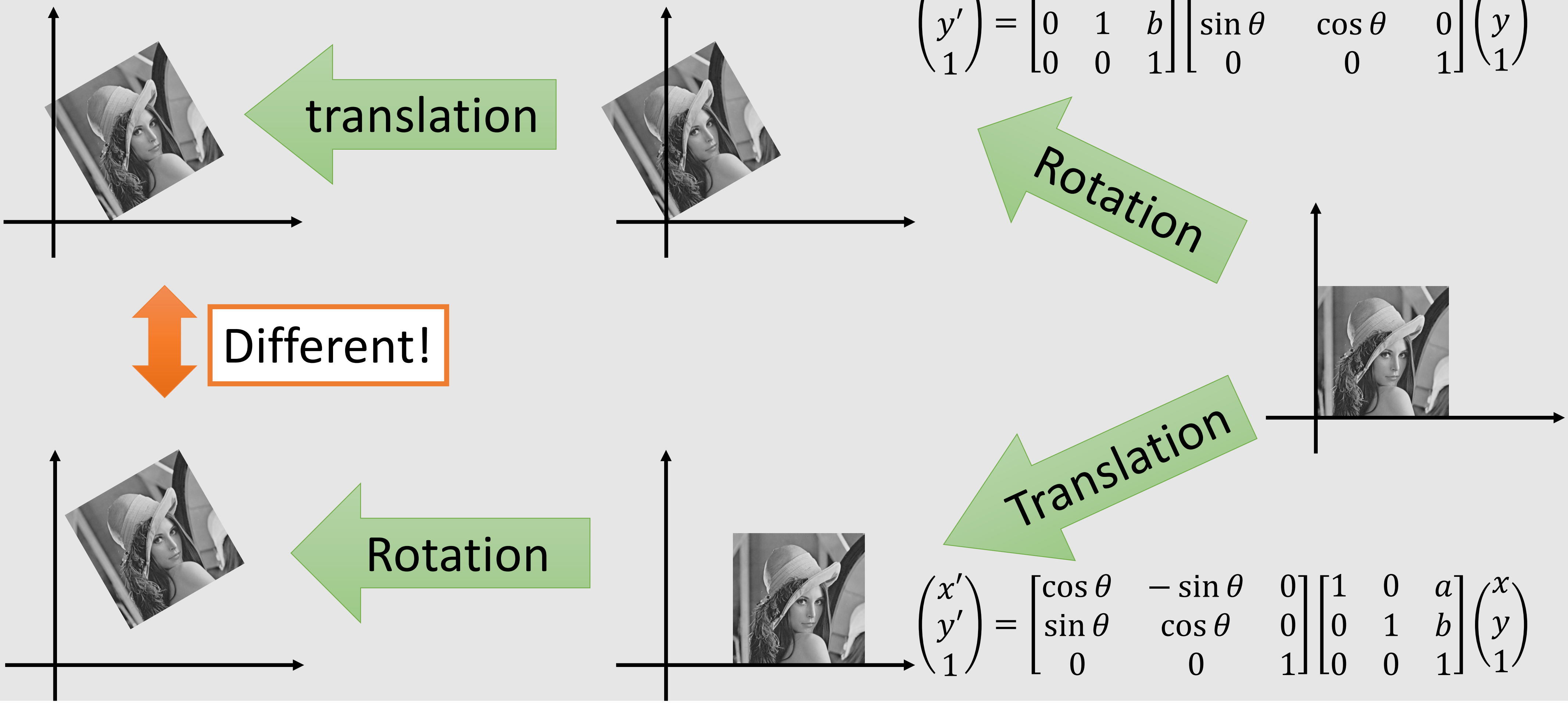
$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Rotation with θ、 then translation with (a, b)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Order Dependency in the Transformation

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

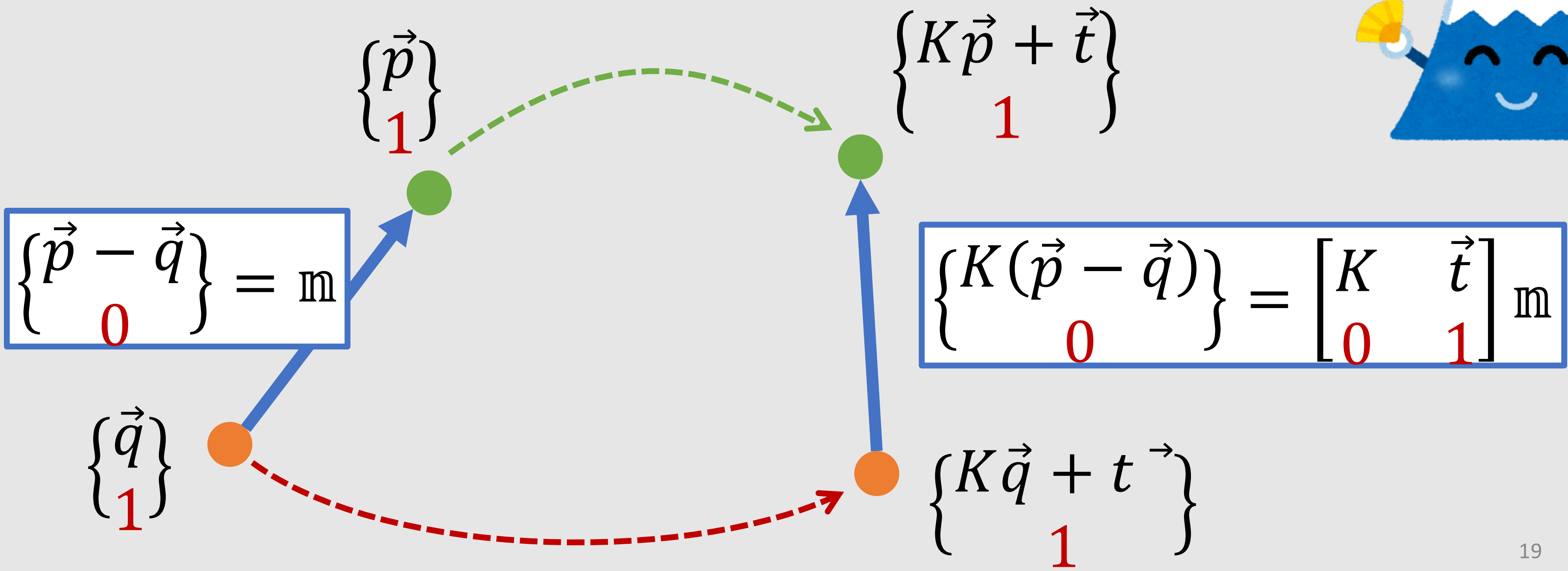


$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Orientation in Affine Transformation

- $\{* \quad * \quad * \quad 1\}^T \rightarrow$ Position
- $\{* \quad * \quad * \quad 0\}^T \rightarrow$ Orientation

This is how the normal vectors are transformed



2D Homographic Transformation

Homogeneous Coordinate in 2D

- Affine transformation uses homogeneous coordinate

Cartesian | Homogeneous

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Some transformation

$$\begin{pmatrix} x'/w \\ y'/w \end{pmatrix} \leftarrow \begin{pmatrix} x' \\ y' \\ w \end{pmatrix}$$

Intuition of 2D Homogeneous Coordinate

- Projecting on the plane (w=1)

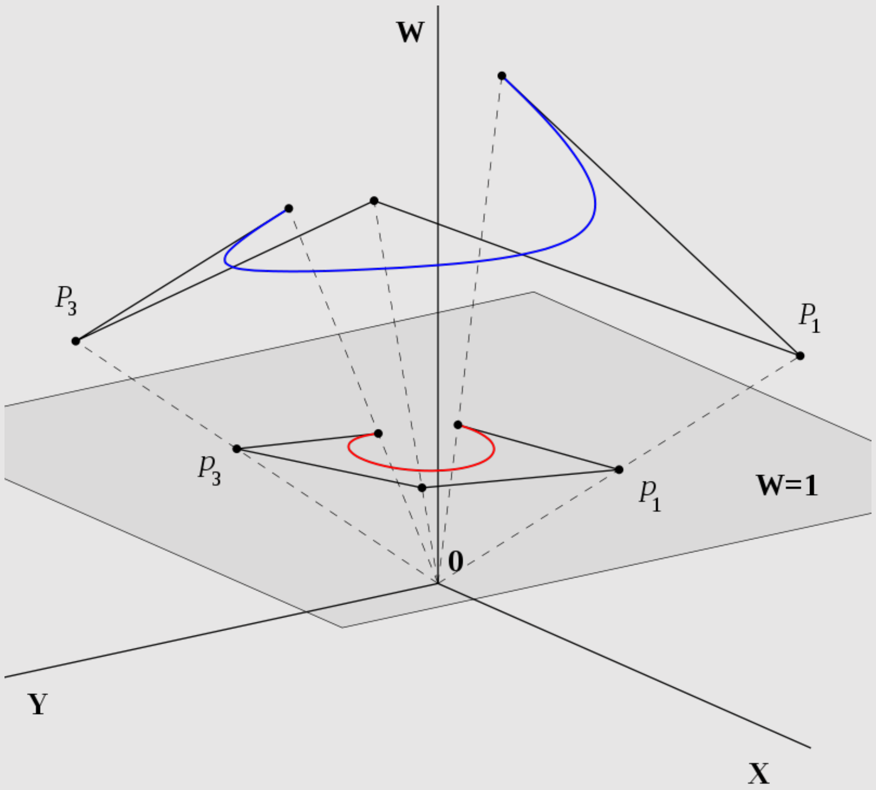
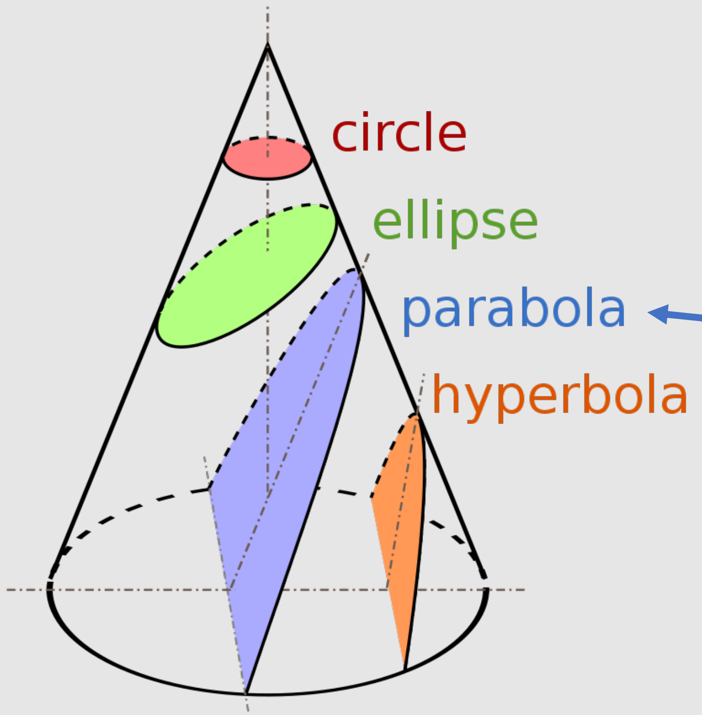


Image Credit: Wojciech Muła @ Wikipedia

Application 1: NURBS 2D Curve

- NURBS=Non-uniform Rational B-Spline
- Projection convert quadratic curve to circle

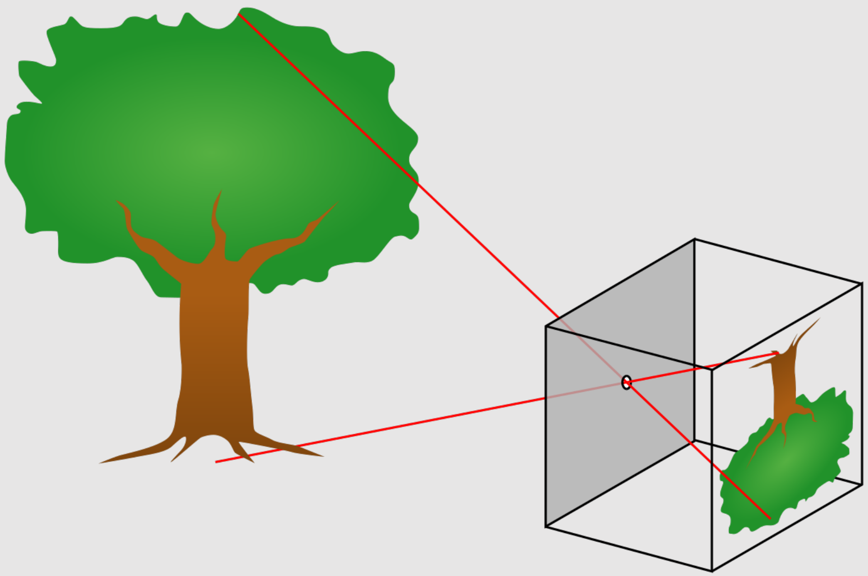


Quadratic curve

Image credit: Magister_Mathematicae @ Wikipedia

Application 2: Perspective Transformation

- Pinhole camera (camera obscura)

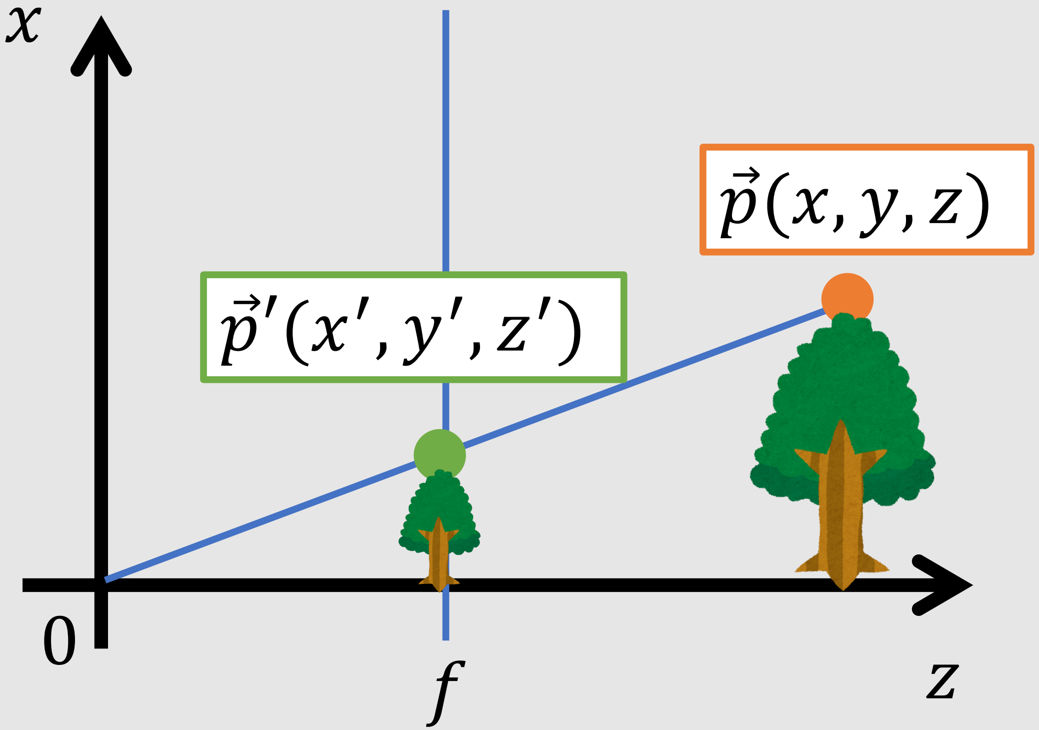


projected image

Image Credit: Gampe @ Wikipedia

Simple Perspective

- Projecting $\vec{p}(x, y, z)$ on the image plane $z = f$ (f: focal length)



$$z' = f, \qquad \frac{x'}{x} = \frac{y'}{y} = \frac{z'}{z}$$

$$x' = \frac{fx}{z}, \qquad y' = \frac{fy}{z}$$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} \propto \begin{pmatrix} x'' \\ y'' \\ w \end{pmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

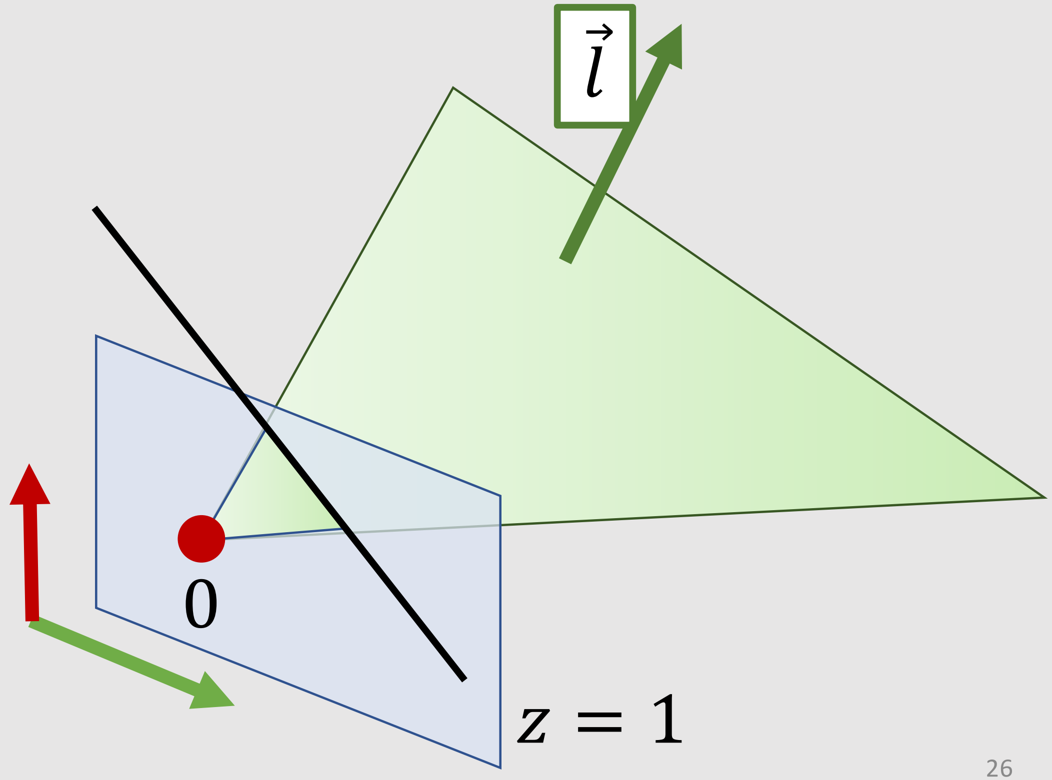
Application 3: 2D Line Equation

- In 2D homogenous coordinate, a dot product defines a line

$$ax + by + c = 0$$

$$\{a \quad b \quad c\} \cdot \begin{Bmatrix} x \\ y \\ 1 \end{Bmatrix} = 0$$

$$\vec{l} \cdot \vec{p} = 0$$



Epipolar Geometry

- Geometry for binocular stereo

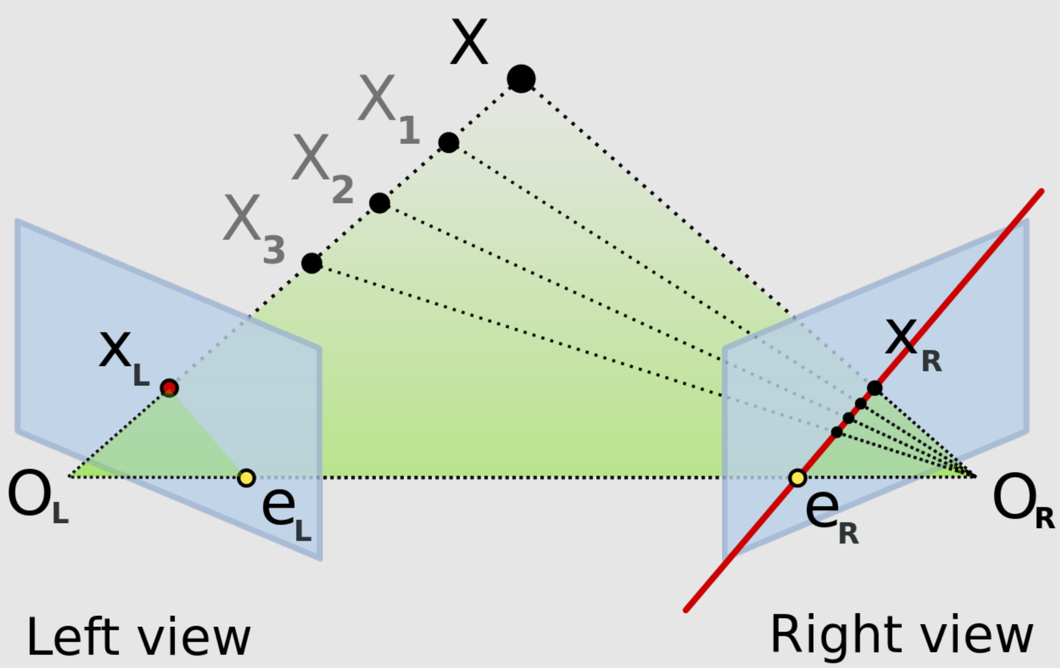


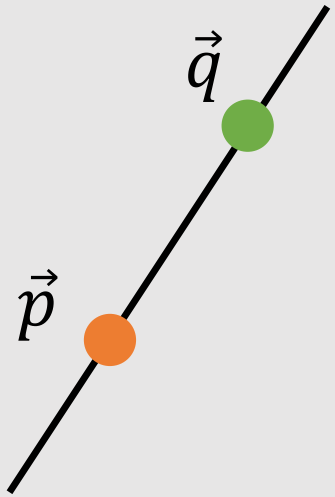
Image Credit: Arne Nordmann @ Wikipedia

This is glasses

This is binoculars

2D Line Passing Through 2 Points

- For a line $\vec{l} \in \mathbb{R}^3$ that passes through two points $\vec{\mathrm{p}}, \vec{q} \in \mathbb{R}^3$



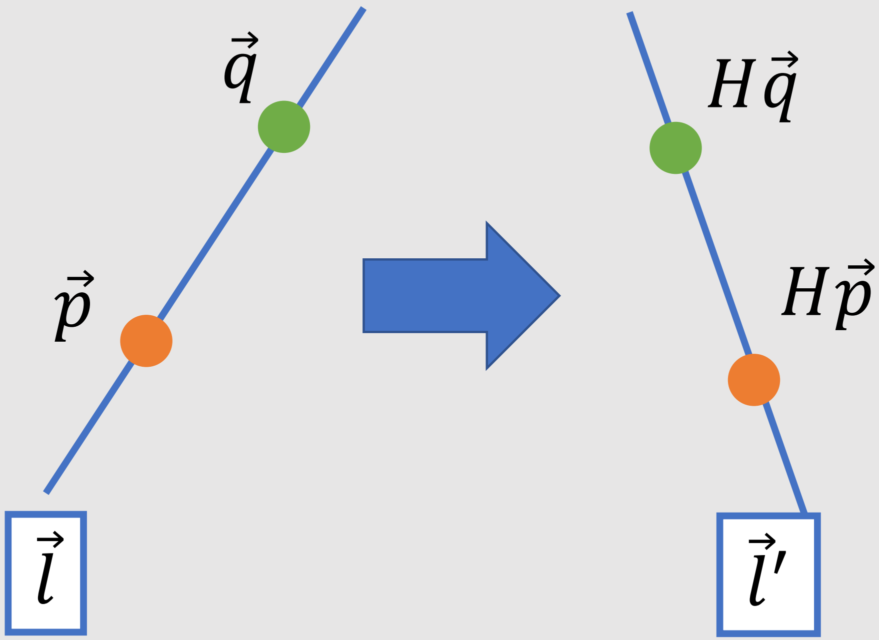
$$\vec{l} \cdot \vec{p} = 0 \quad \text{and } \vec{l} \cdot \vec{q} = 0$$

$$\vec{l} = \vec{p} \times \vec{q}$$

$$= \begin{pmatrix} p_2 q_3 - p_3 q_2 \\ p_3 q_1 - p_1 q_3 \\ p_1 q_2 - p_2 q_1 \end{pmatrix} = \begin{pmatrix} p_y - q_y \\ q_x - p_x \\ p_x q_y - p_y q_x \end{pmatrix}$$

Transformation of Line

- How the line is transformed?



$$\vec{l}' \cdot (H\vec{p}) = 0 \text{ and } \vec{l}' \cdot (H\vec{q}) = 0$$

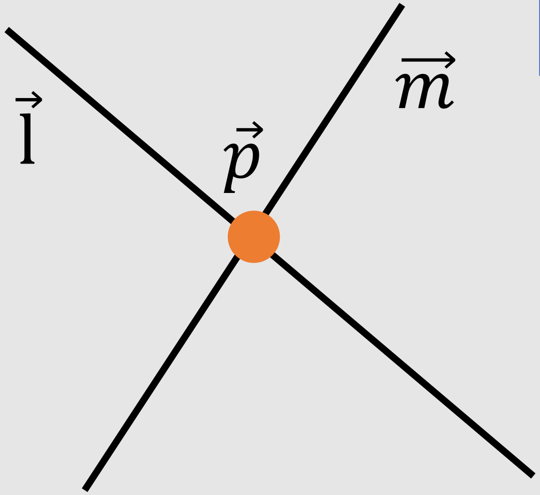
$$\left(H^T\vec{l}'\right) \cdot \vec{p} = 0 \text{ and } \left(H^T\vec{l}'\right) \cdot \vec{q} = 0$$

$$H^T\vec{l}' = \vec{\mathrm{p}} \times \vec{q} = \vec{l}$$

$$\vec{l}' = H^{-T}\vec{l}$$

Intersection of 2 Lines

- Intersection $\vec{p} \in \mathbb{R}^3$ of two lines $\vec{l}, \vec{m} \in \mathbb{R}^3$

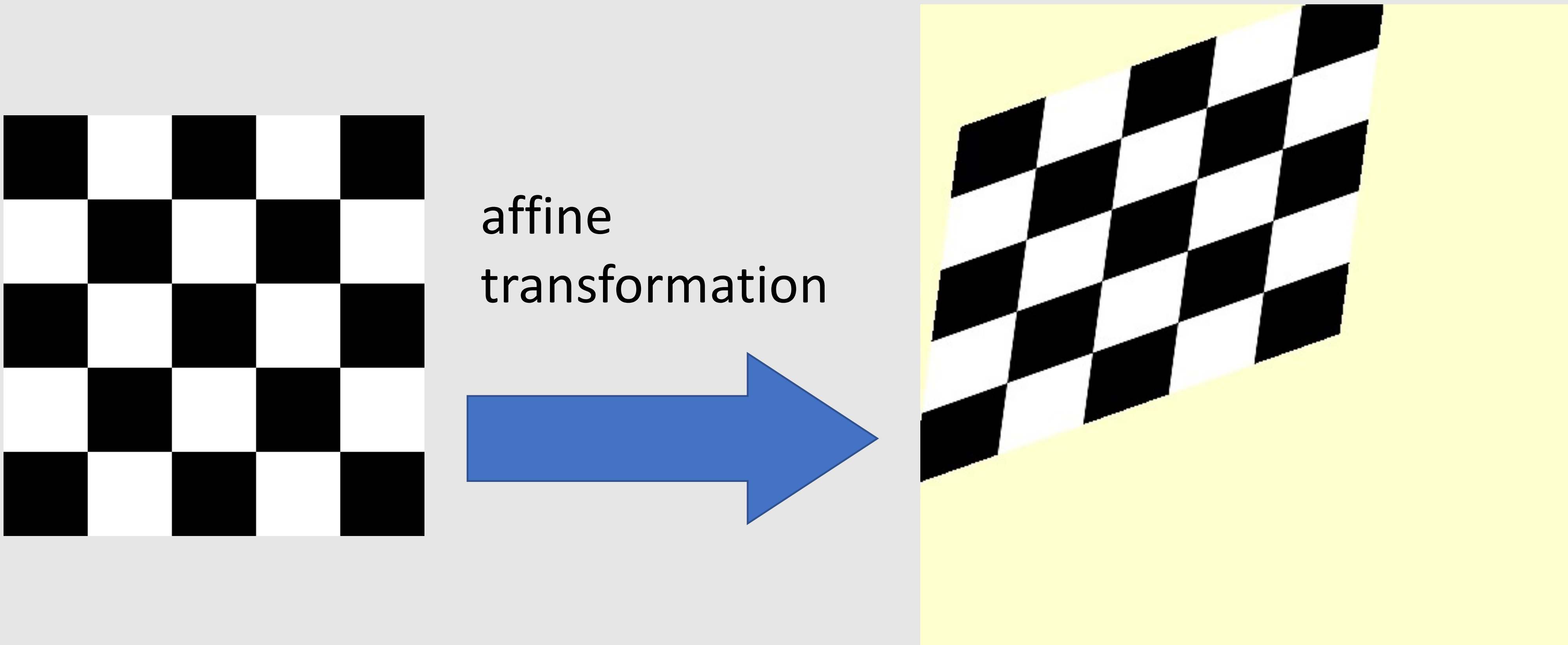


$$\vec{l} \cdot \vec{p} = 0 \quad \text{and} \quad \vec{m} \cdot \vec{p} = 0$$

$$\vec{p} = \vec{l} \times \vec{m} = \begin{pmatrix} l_2 m_3 - l_3 m_2 \\ l_3 m_1 - l_1 m_3 \\ l_1 m_2 - l_2 m_1 \end{pmatrix}$$

What if $\vec{m}$ and $\vec{l}$ are parallel

Affine Transformation: Preserve Parallelism

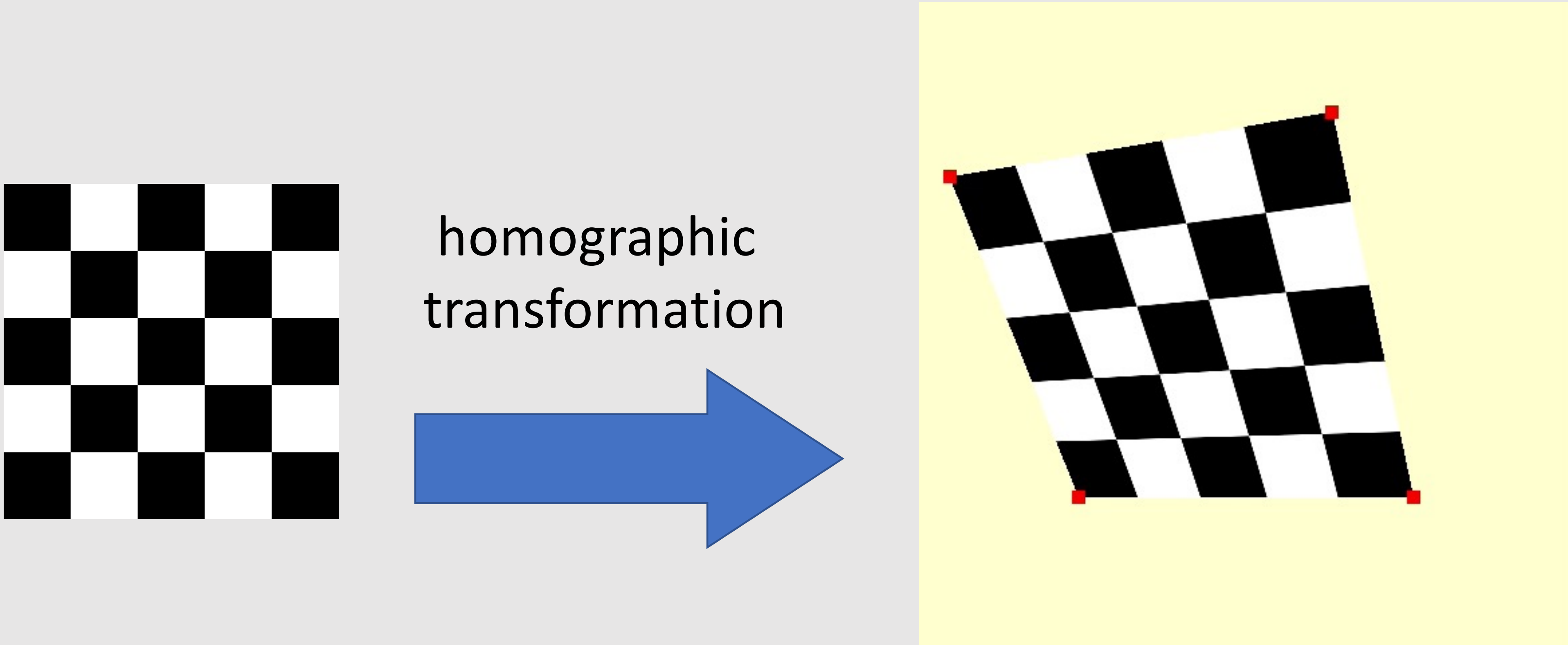


How Affine Preserve Parallerism?

- Parallel lines -> intersect at infinity -> intersect at $\vec{p} = \{*,*,0\}^T$
- Transformed intersection $H\vec{p}$ also need to intersect at infinity

$$H\vec{p} = \begin{bmatrix} * & * & * \\ * & * & * \\ ? & ? & 1 \end{bmatrix} \begin{Bmatrix} * \\ * \\ 0 \end{Bmatrix} = \begin{Bmatrix} * \\ * \\ 0 \end{Bmatrix}$$

Homographic Transformation: Preserve Line

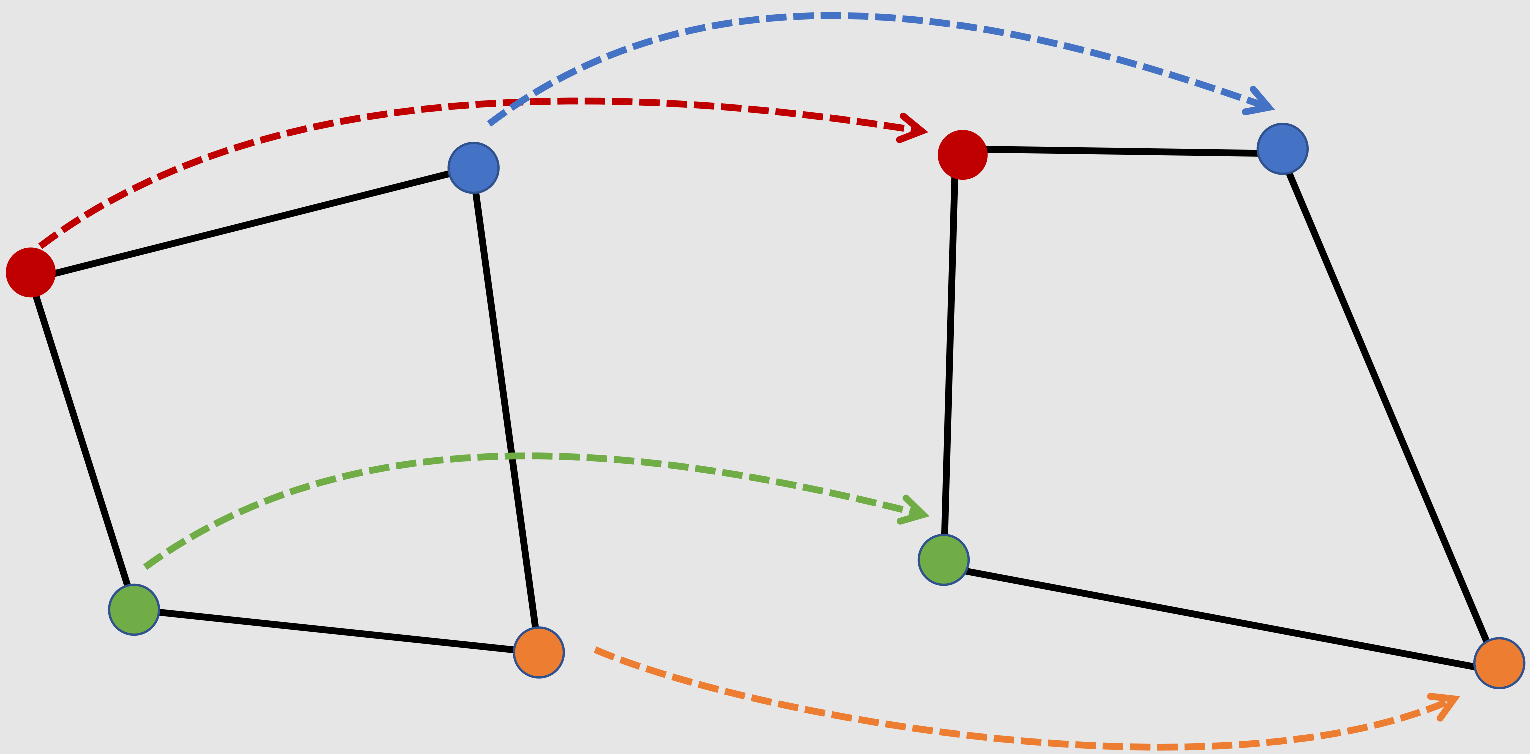


How the Homography Preserve Line?

- Show that if points $\vec{p}$ lie on some line $\vec{l}$,
  then their transformed points $\vec{p}' = H\vec{p}$ also lie on some line $\vec{l}'$

Homography: Mapping 4 Points

- How the mapping of 4 points uniquely define 3x3 Homography matrix?



Homographic Transformation of 4 Points

$$\begin{bmatrix} a_0 & a_1 & a_2 \\ a_3 & a_4 & a_5 \\ a_6 & a_7 & 1 \end{bmatrix} \begin{Bmatrix} c_{0x} & c_{1x} & c_{2x} & c_{3x} \\ c_{0y} & c_{1y} & c_{2y} & c_{4y} \\ 1 & 1 & 1 & 1 \end{Bmatrix} \rightarrow \begin{Bmatrix} d_{0x} & d_{1x} & d_{2x} & d_{3x} \\ d_{0y} & d_{1y} & d_{2y} & d_{4y} \\ 1 & 1 & 1 & 1 \end{Bmatrix}$$

$$\frac{a_0 c_{ix} + a_1 c_{iy} + a_2}{a_6 c_{ix} + a_7 c_{iy} + 1} = d_{ix}$$

How many points for 3D?

8 degrees of freedom (DoFs)
8 constraints!

Transformation & Conservation

- Homographic transformation -> Preserve line
- Affine transformation -> Preserve parallelism
- Linear transformation -> Preserve origin, line, parallelism
- Rigid transformation -> Preserve length, volume
- Conformal transformation -> Preserve angle

Conformal Transformation

- 2D transformation that doesn't change angles
- functions in complex plane

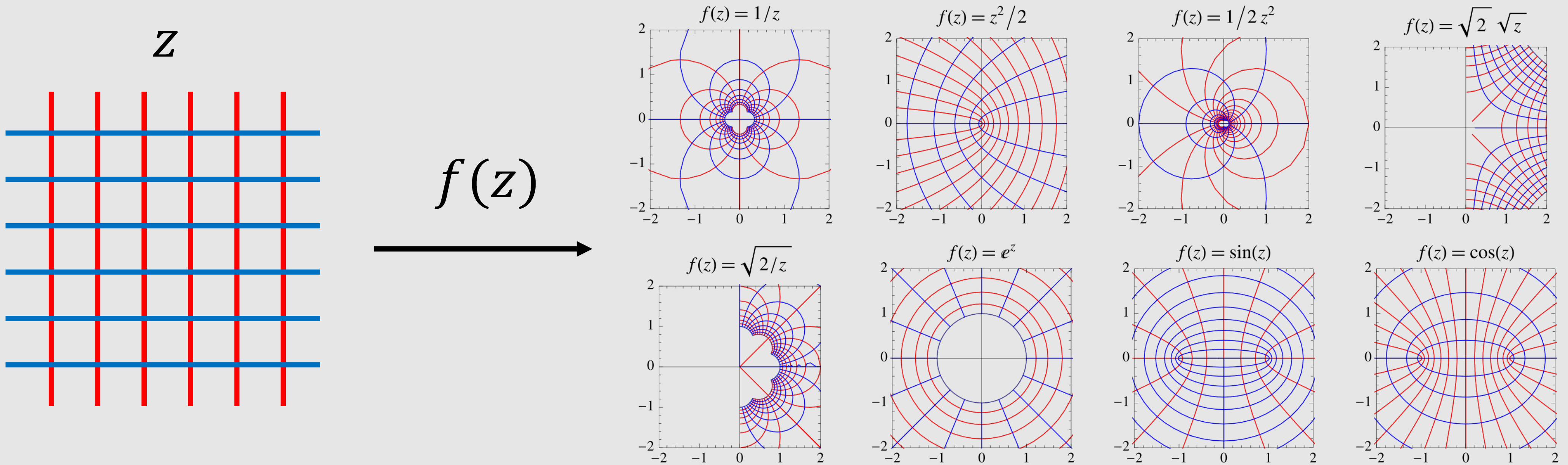
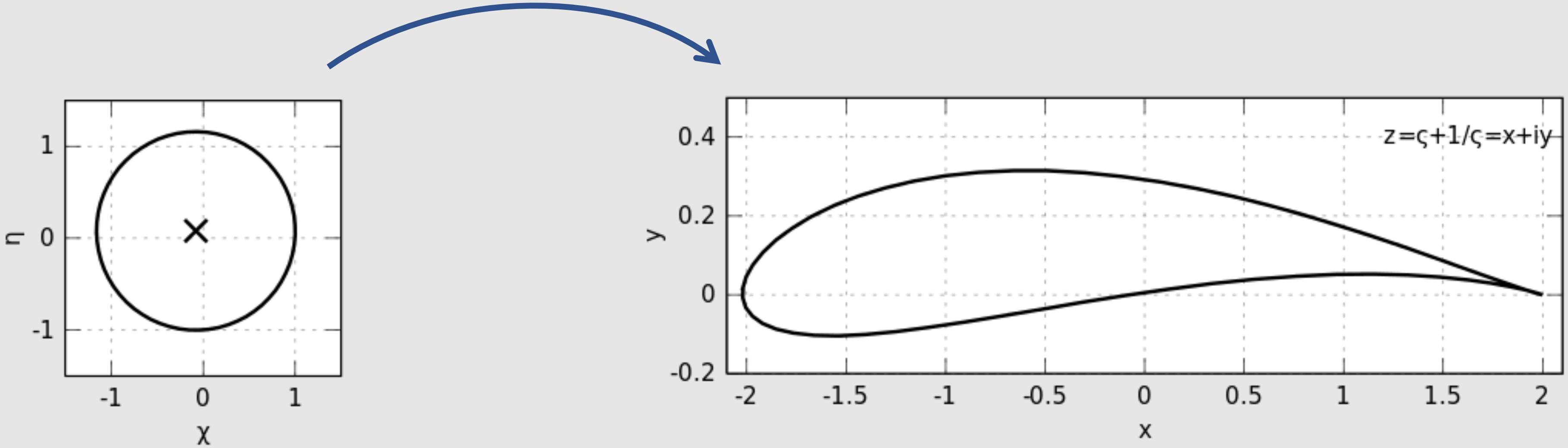


Image Credit: Wolfram Mathworld https://mathworld.wolfram.com/ConformalMapping.html

Comformal Transformation & Engineering

- Joukowsky airfoil

$$f(z) = z + \frac{1}{z}$$



Images from https://www.wikiwand.com/en/Joukowsky_transform

Conformal Transformation & Biology

- D'Arcy Thompson's theory

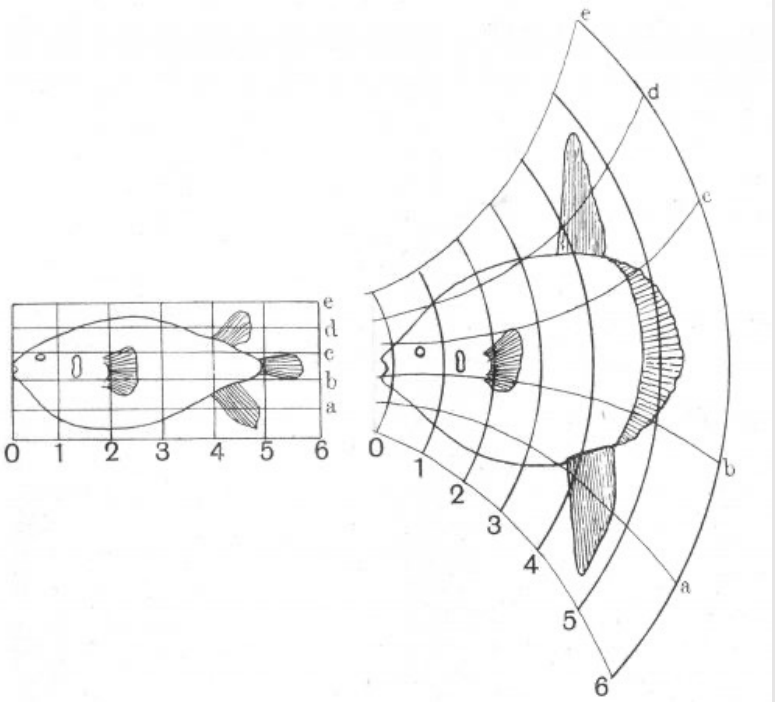
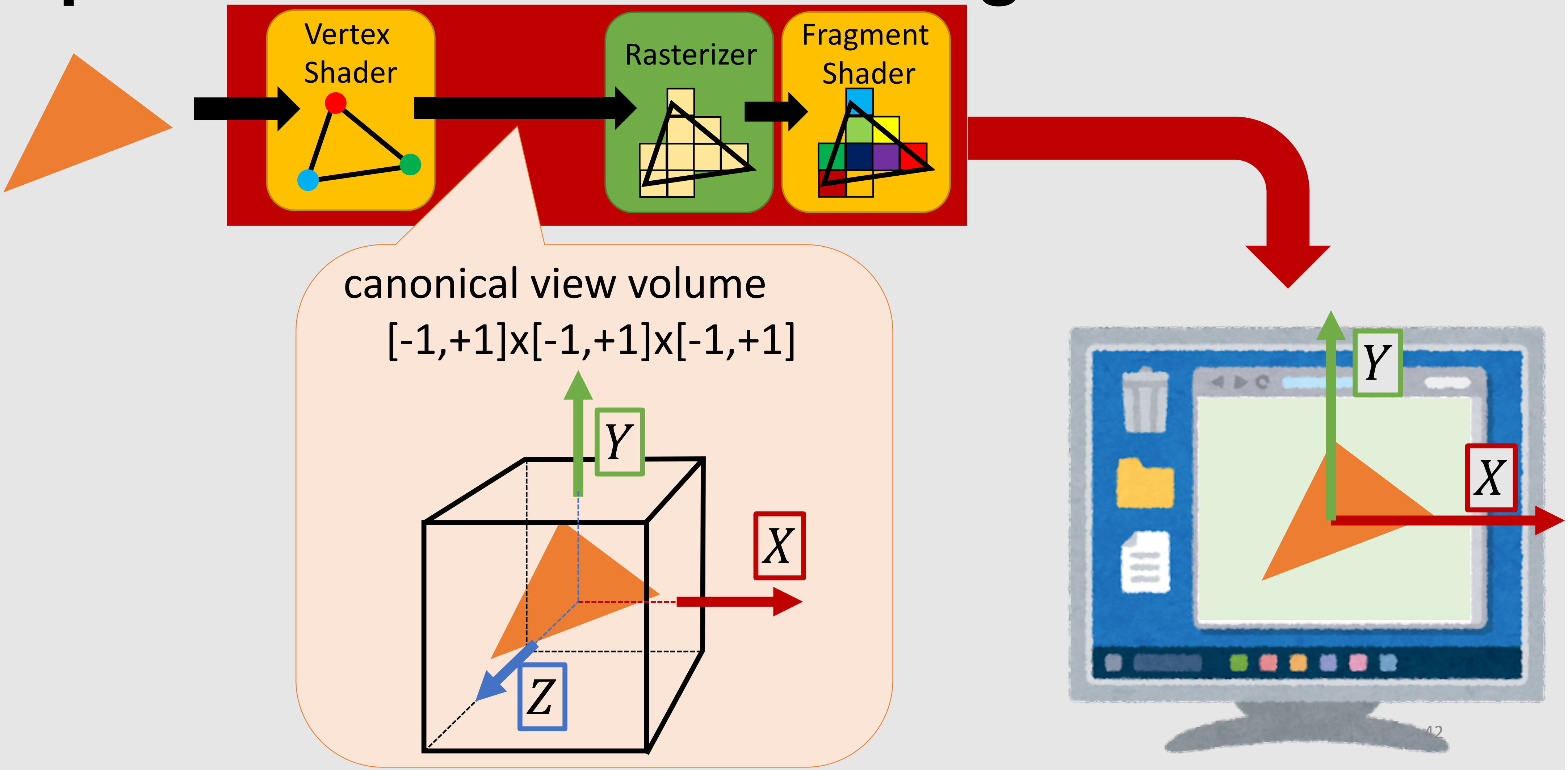


Fig. 381. *Diodon.* Fig. 382. *Orthagoriscus.*

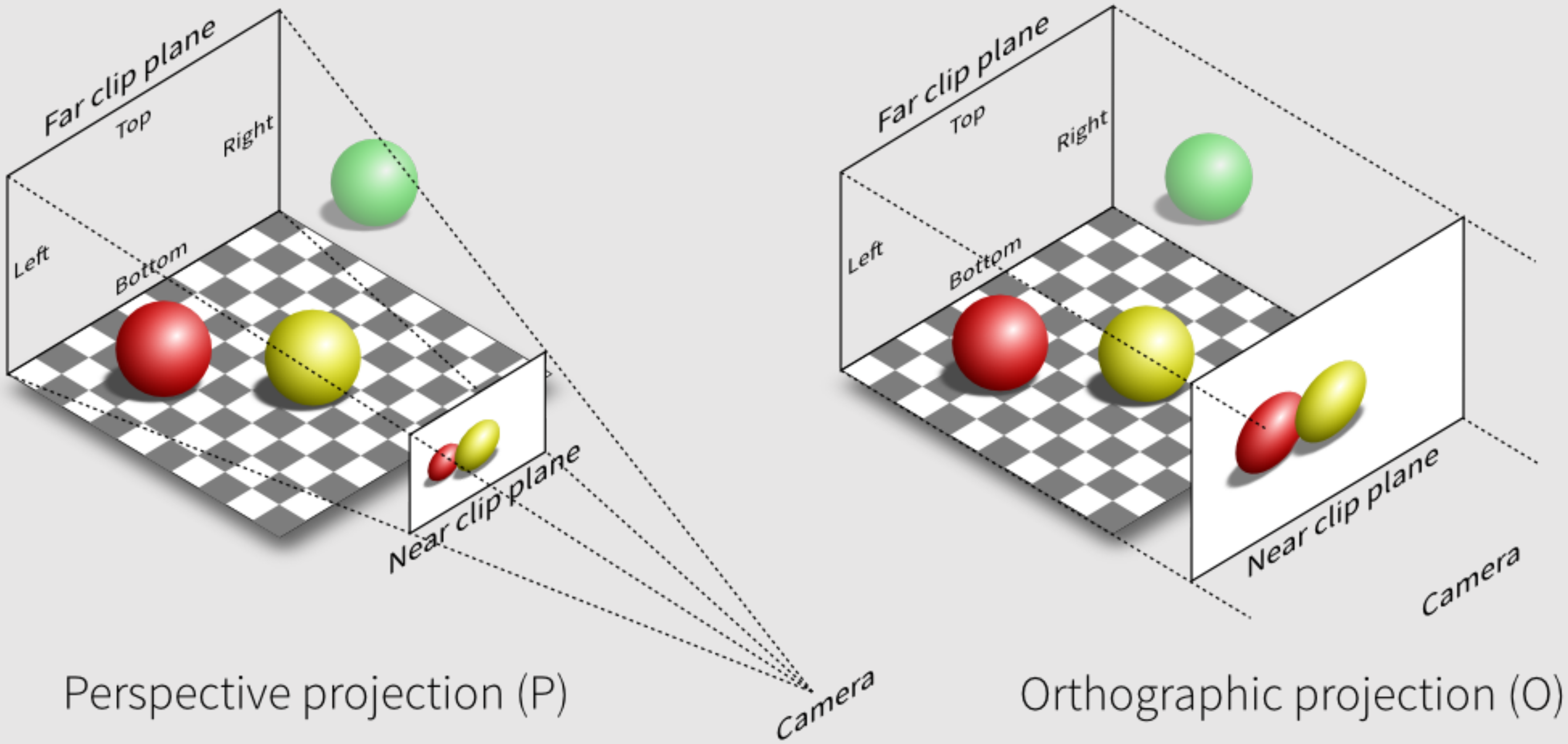
Thompson, D'A. W. *On Growth and Form* (Cambridge Univ. Press, Cambridge, 1917)

View Transformation for 3D Graphics

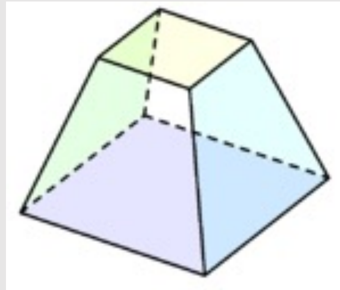
Input of the Rasterizer: Triangle in a Cube



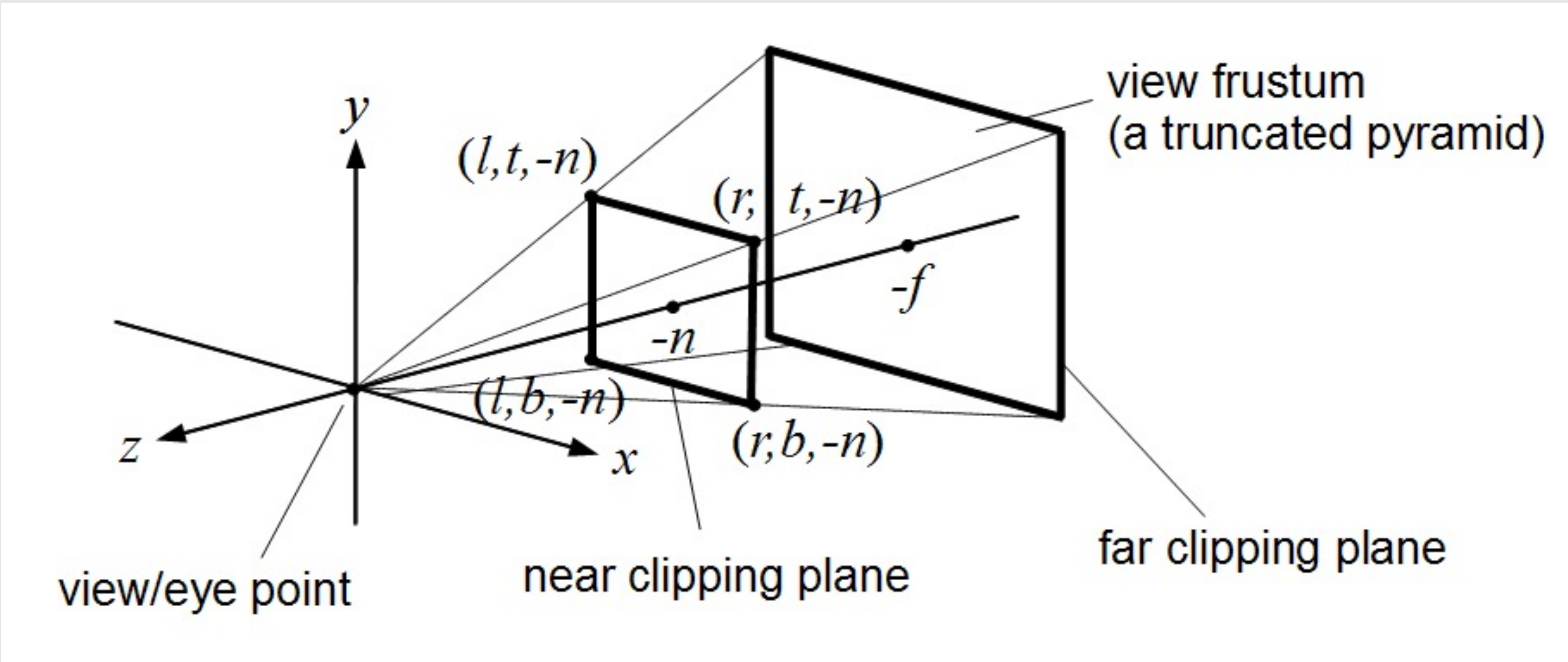
Perspective & Ortographic Projection



View Frustum（視錐台）



- View frustum is region the camera see



Pinhole Camera Model