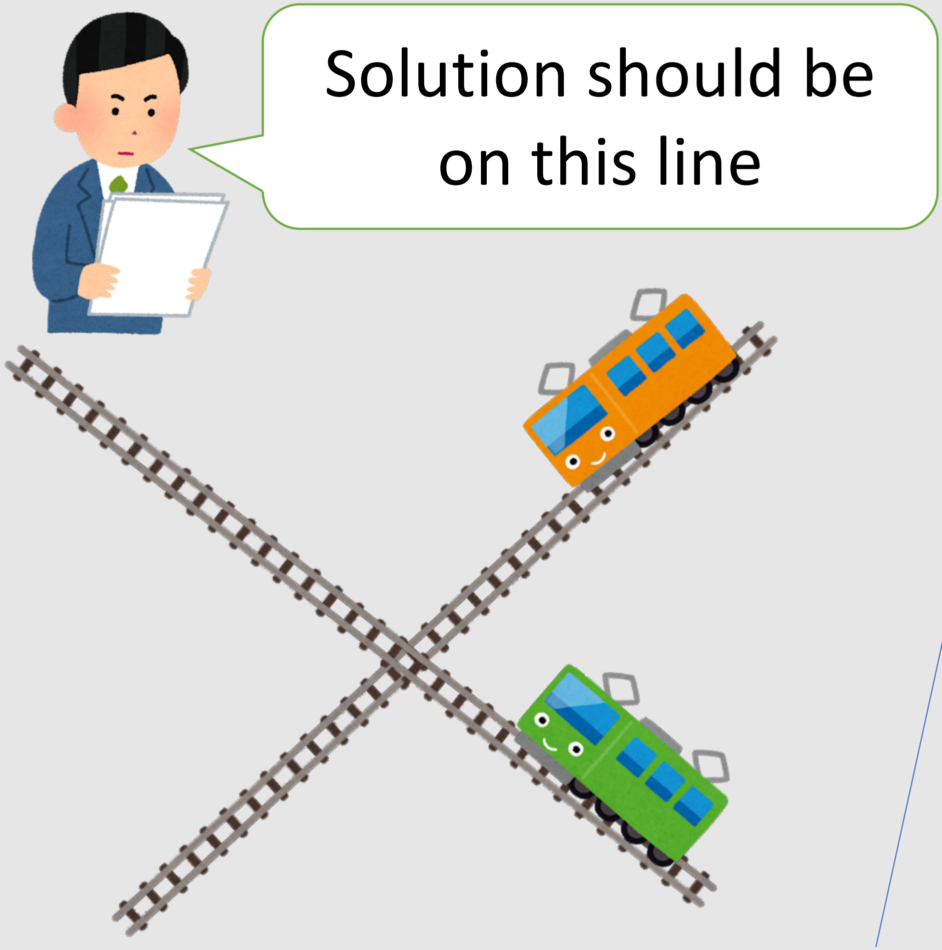
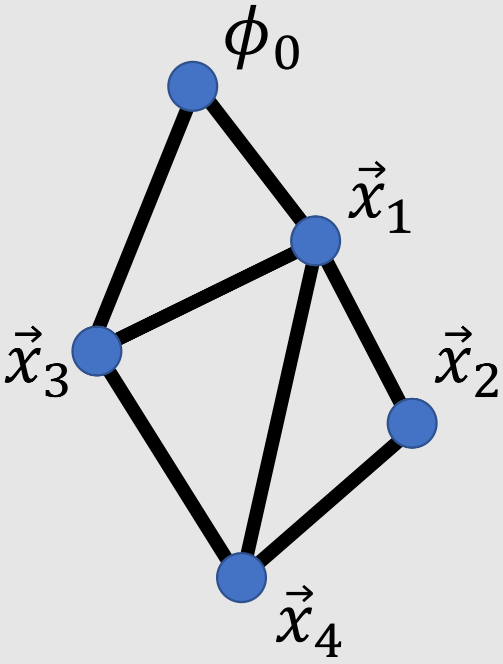
Solving Constraints v.s. Optimization



Graph Laplacian Matrix as Constraints

- $L\vec{v} = 0$ means all the vertices are average of connected ones

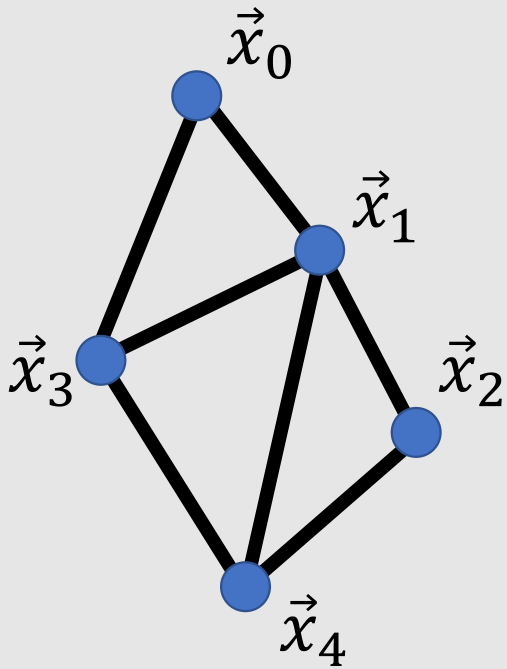


$$L\Phi = 0$$

$$\Longrightarrow \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{bmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{pmatrix} = 0$$

Graph Laplacian Matrix as Optimization

- $L\Phi = 0$ means sum of square difference is minimized

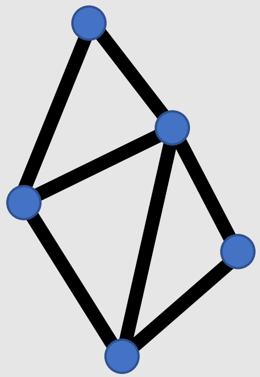


$$W = \frac{1}{2}\sum_{e \subset \mathcal{E}} \left\| v_{e_1} - v_{e_2} \right\|^2$$

$$= \frac{1}{2}\vec{v}^T L \vec{v}$$

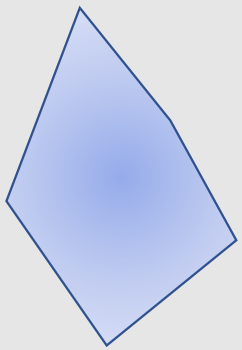
W is minimized $\rightarrow \frac{\partial W}{\partial \vec{v}} = L\vec{v} = 0$

Laplacian in Continum Domain



$$L\Phi = 0$$

$$W = \sum_{e \subset \mathcal{E}} \left\| \phi_{e_1} - \phi_{e_2} \right\|^2 = \Phi^{\mathrm{T}} L \Phi$$



$$\nabla \cdot \nabla \phi = 0$$

$$W = \int_{\Omega} \|\nabla \phi\|^2 dV$$

Dirichelet energy!

Partial Differential Equation (PDE)

Nabla Operator

Nabla: $\nabla = \vec{e}_x \frac{\partial}{dx} + \vec{e}_y \frac{\partial}{dy} + \vec{e}_z \frac{\partial}{dz}$

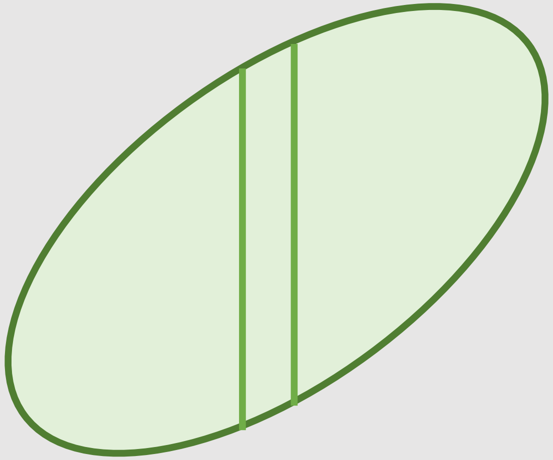
Gradient: $\nabla \phi = \vec{e}_x \frac{\partial \phi}{dx} + \vec{e}_y \frac{\partial \phi}{dy} + \vec{e}_z \frac{\partial \phi}{dz}$

Divergence: $\nabla \cdot \vec{v} = \frac{\partial v_x}{dx} + \frac{\partial v_y}{dy} + \frac{\partial v_z}{dz}$

Gauss Divergence Theorem

- Convert volume integration to surface integration

$$\int_{\Omega} \nabla \cdot \vec{v}\, d\mathrm{V} = \int_{\partial\Omega} \vec{n} \cdot \vec{v}\, dS$$

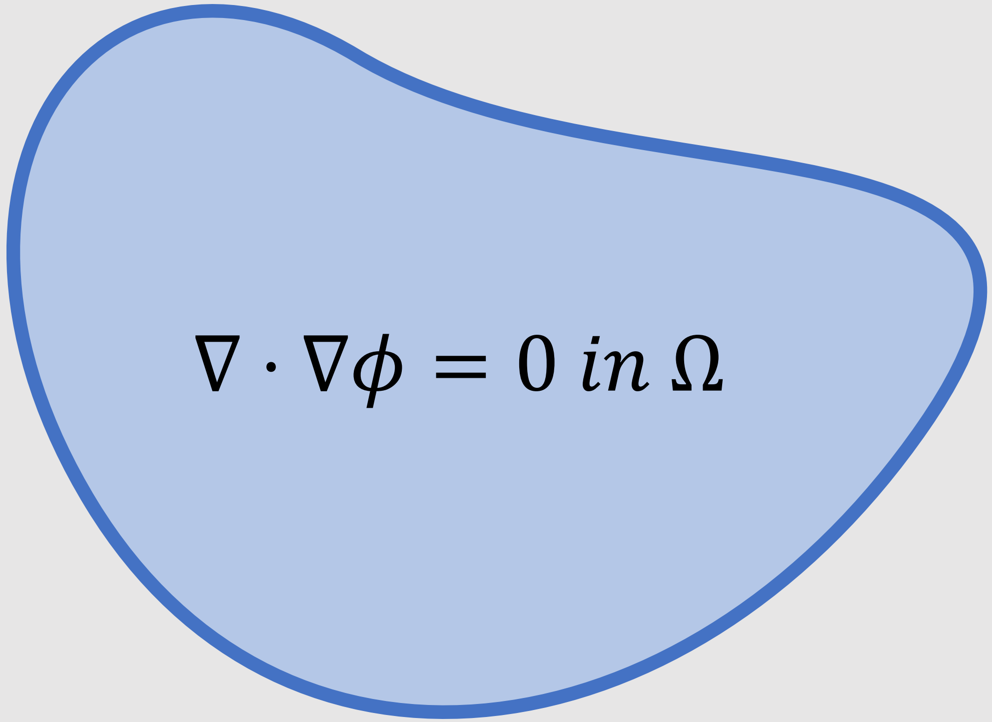


Chain Rule of Nabla Operator

$$\nabla \cdot (\phi \vec{v}) = (\nabla \phi)^T \vec{v} + \phi (\nabla \cdot \vec{v})$$

Laplace Equation

$$\nabla \cdot \nabla \phi = 0$$



$$\phi = \phi_0 \ on \ \partial\Omega$$

Finite Difference Method

- Approximate PDE with differences

$$\nabla \cdot \nabla \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Solution with Finite Element Method

- Solution of Laplace equation minimize Dirichlet energy

$$\nabla \cdot \nabla \phi \ = 0$$

Dirichlet energy

$$W = \int \|\nabla \phi\|^2 d\Omega$$

Solution with Finite Boundary Method

- Represent solution with the fundermental solution of Laplacian

$$\nabla \cdot \nabla \phi = 0$$

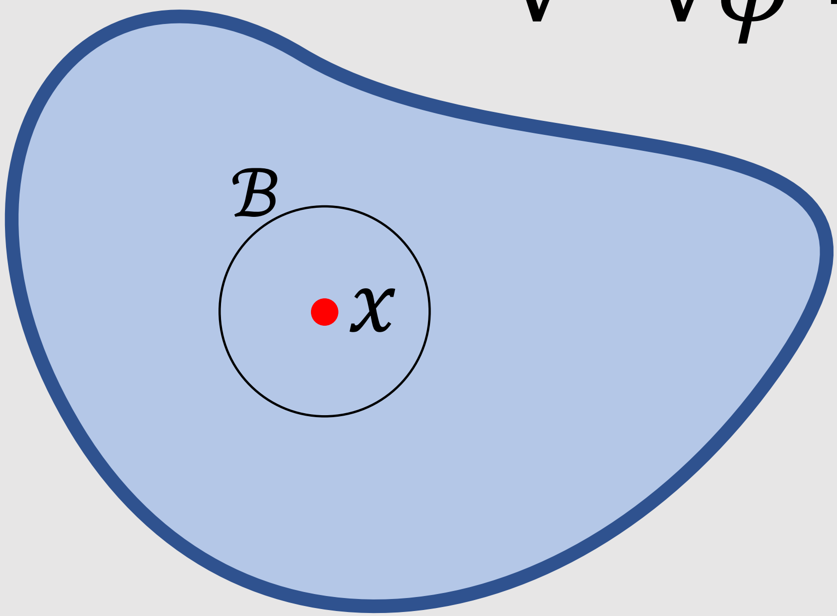
fundermental solution

$$\nabla \cdot \nabla \phi = \delta(x) \Rightarrow \begin{cases} \phi = |x| & \text{1 dim.} \\ \phi = \frac{1}{2\pi}\log|x| & \text{2 dim.} \\ \phi = -\frac{1}{4\pi|x|} & \text{3 dim.} \end{cases}$$

Solution with Mean Value Theorem

- Mean value theorem: solution is average of the value on the small sphere

$$\nabla \cdot \nabla \phi = 0 \Rightarrow \phi(x) = \frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} \phi\,(y) dy$$

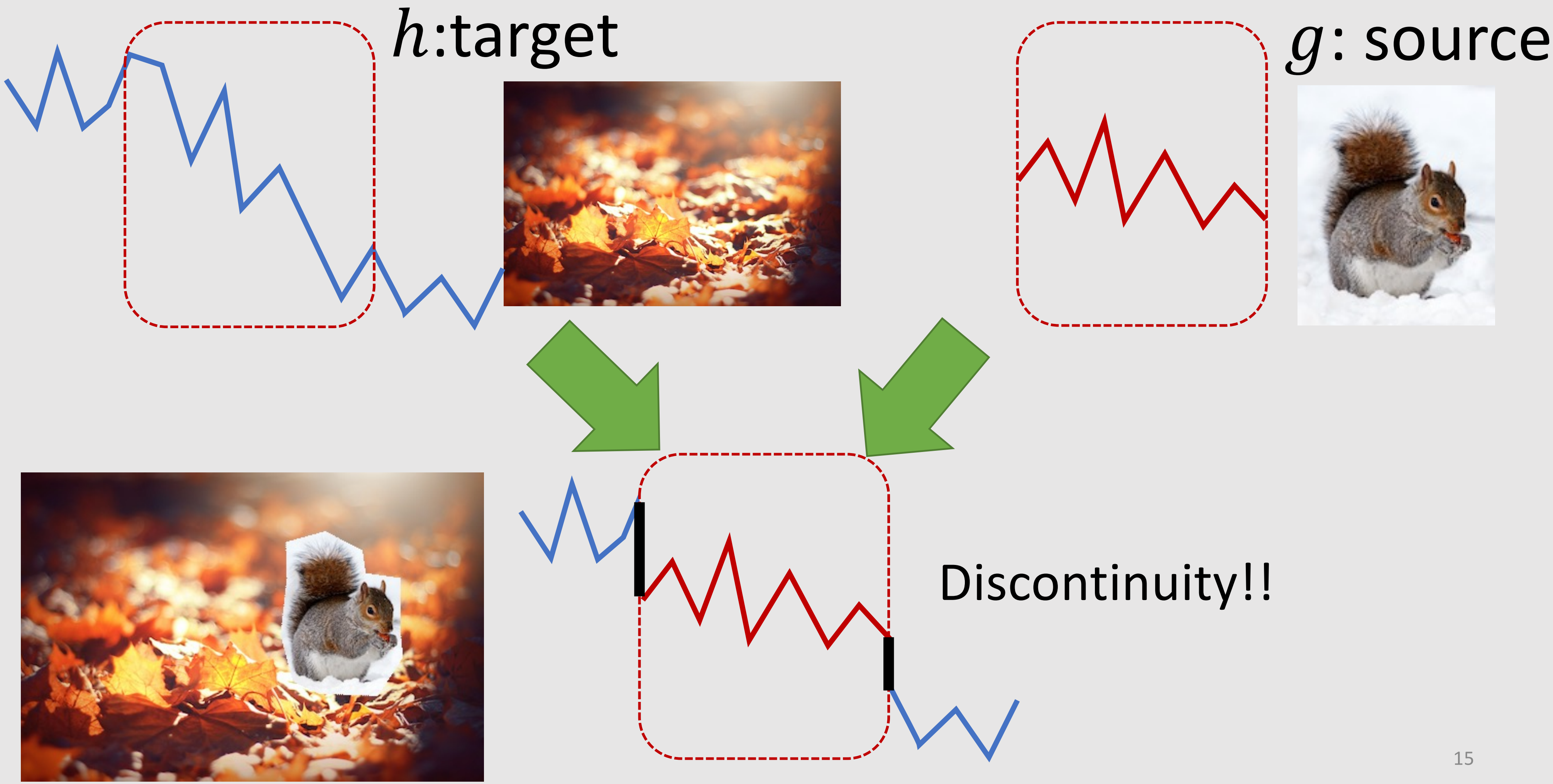


$\phi = \phi_0 \; on \; \partial\Omega$

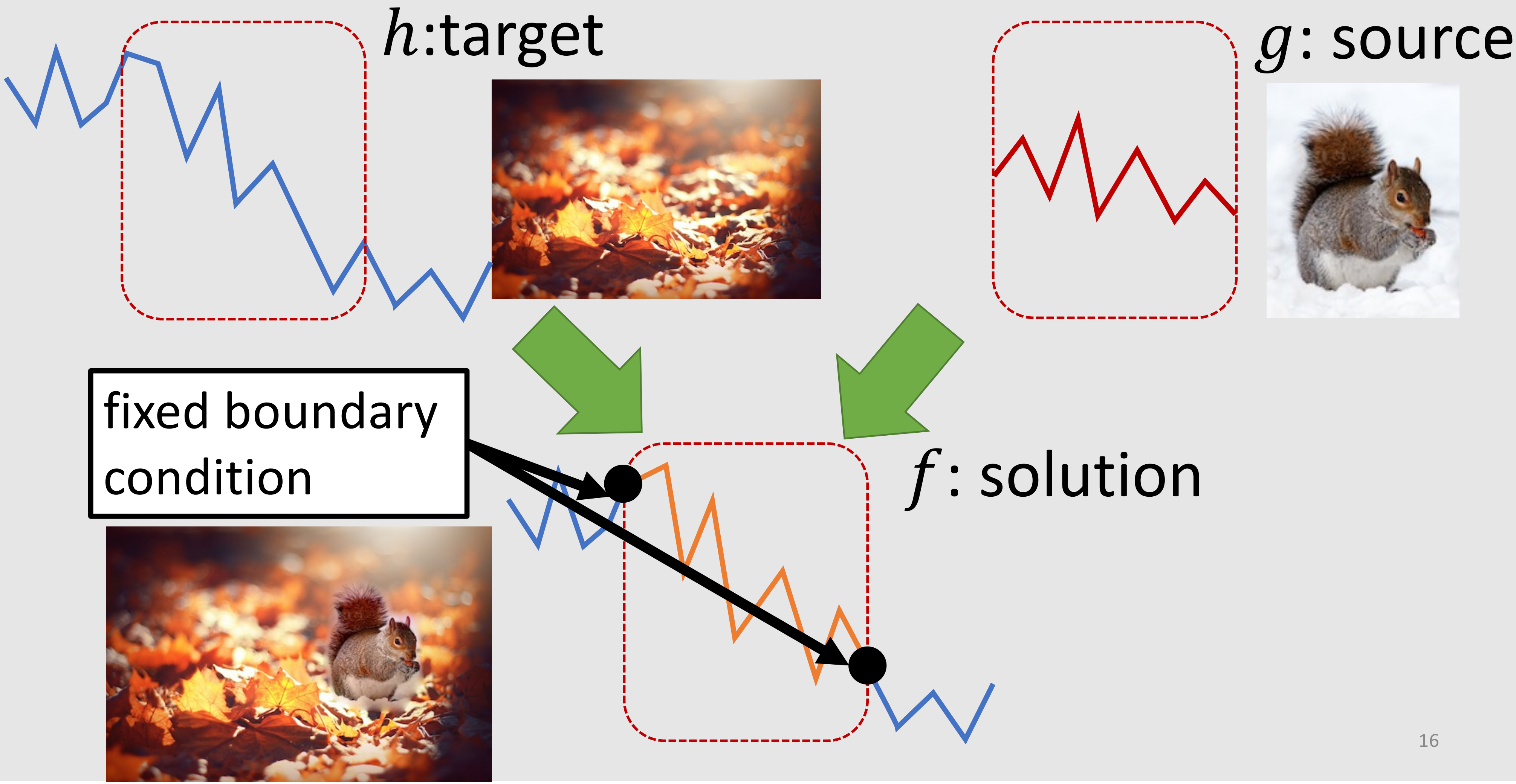
- Mean value coordinate
- Walk-on-sphere method

Poisson Image Editing

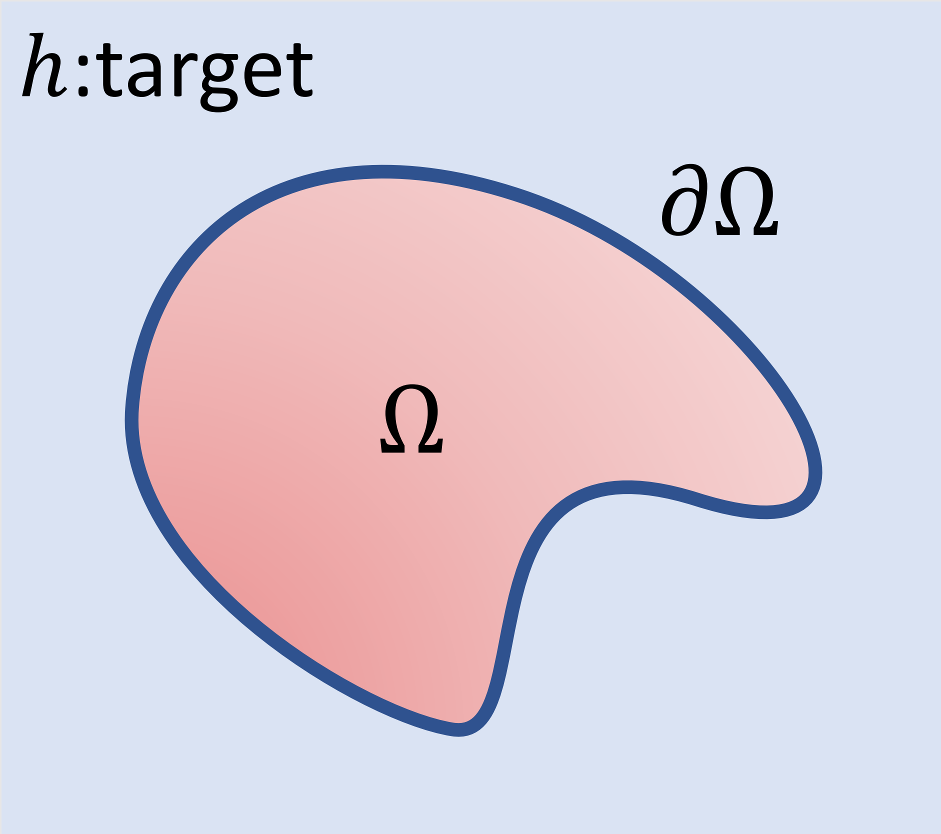
Naïve Blending (1D)



Gradient Domain Blending (1D)



Gradient Domain Blending (2D)



Weak Form of PDE

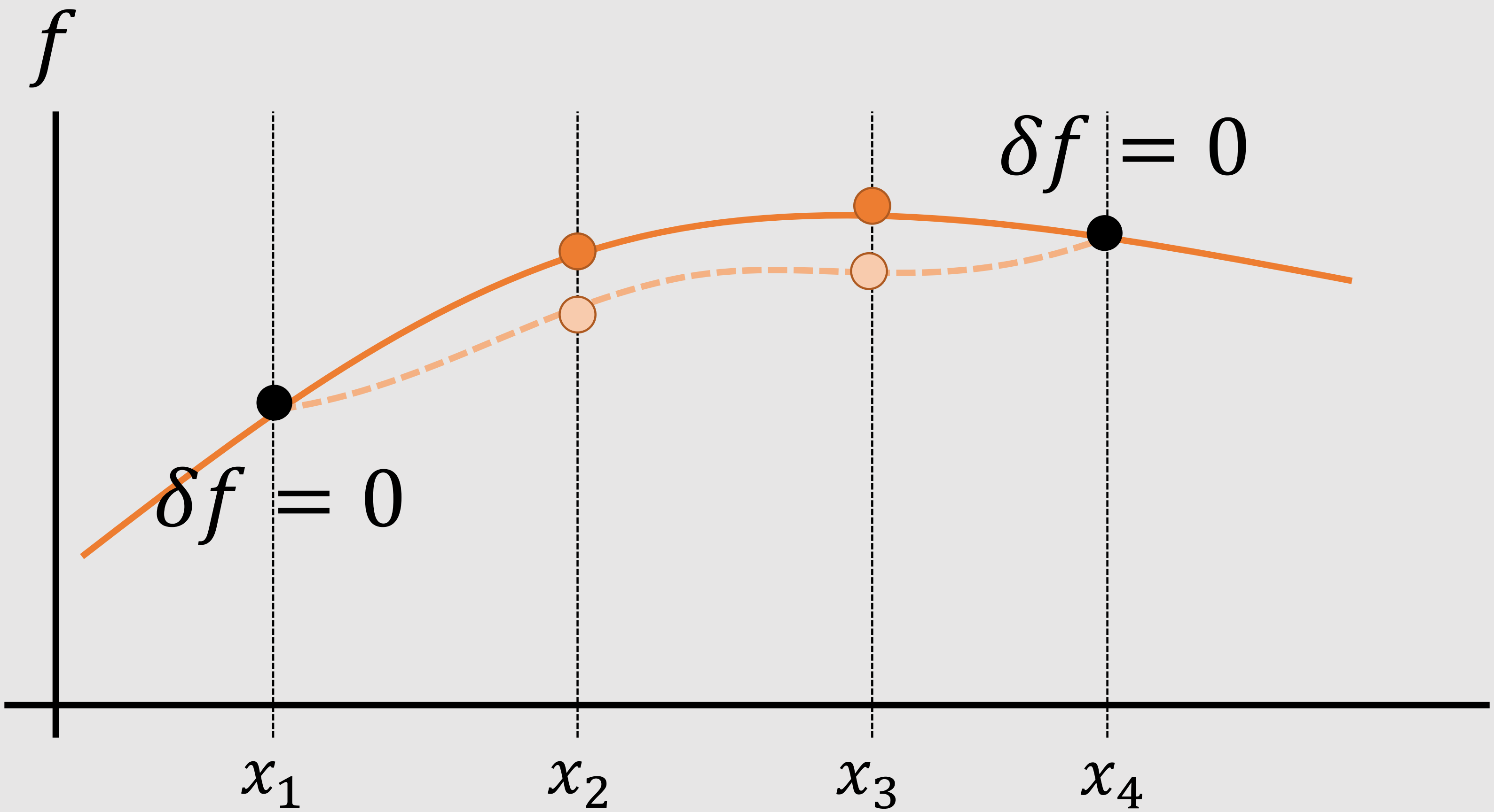
$$W(f) = \int_{\Omega} \|\nabla(f - g)\|^2 \, dV \qquad \bar{f} = \underset{f}{\operatorname{argmin}}\, W(f)$$

Poisson's equation

$$\nabla \cdot \nabla \bar{f} = \nabla \cdot \nabla g$$

Fixed

Purturbation of Solution



Weak Form of PDE

$$W(f) = \int_\Omega \|\nabla f - \nabla g\|^2 \, d\mathrm{V} \qquad \bar{f} = \underset{f}{\operatorname{argmin}} \, W(f)$$

$$\delta W(f, \delta f) = \int_\Omega \{\nabla(f + \delta f) - \delta g\}^T \{\nabla(f + \delta f) - \nabla g\} d\mathrm{V} - W(f)$$

$$= 2 \int_\Omega (\nabla \delta f)^{\mathrm{T}} \nabla (f - g) dV$$

$$\delta W\left(\bar{f}, \delta f\right) = 0, \forall \delta f$$

$$\nabla \cdot (\phi \vec{v}) = (\nabla \phi)^T \vec{v} + \phi(\nabla \cdot \vec{v})$$

$$\phi = \delta f, \vec{v} = f - g$$

$$\nabla \cdot \{\delta f \nabla (f-g)\} = (\nabla \delta f)^T \nabla (f-g) + \delta f \{\nabla \cdot \nabla (f-g)\}$$

$$\int$$

Gauss-Seidel Method

- Solve & update solution $\boldsymbol{x}$ row-by-row

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

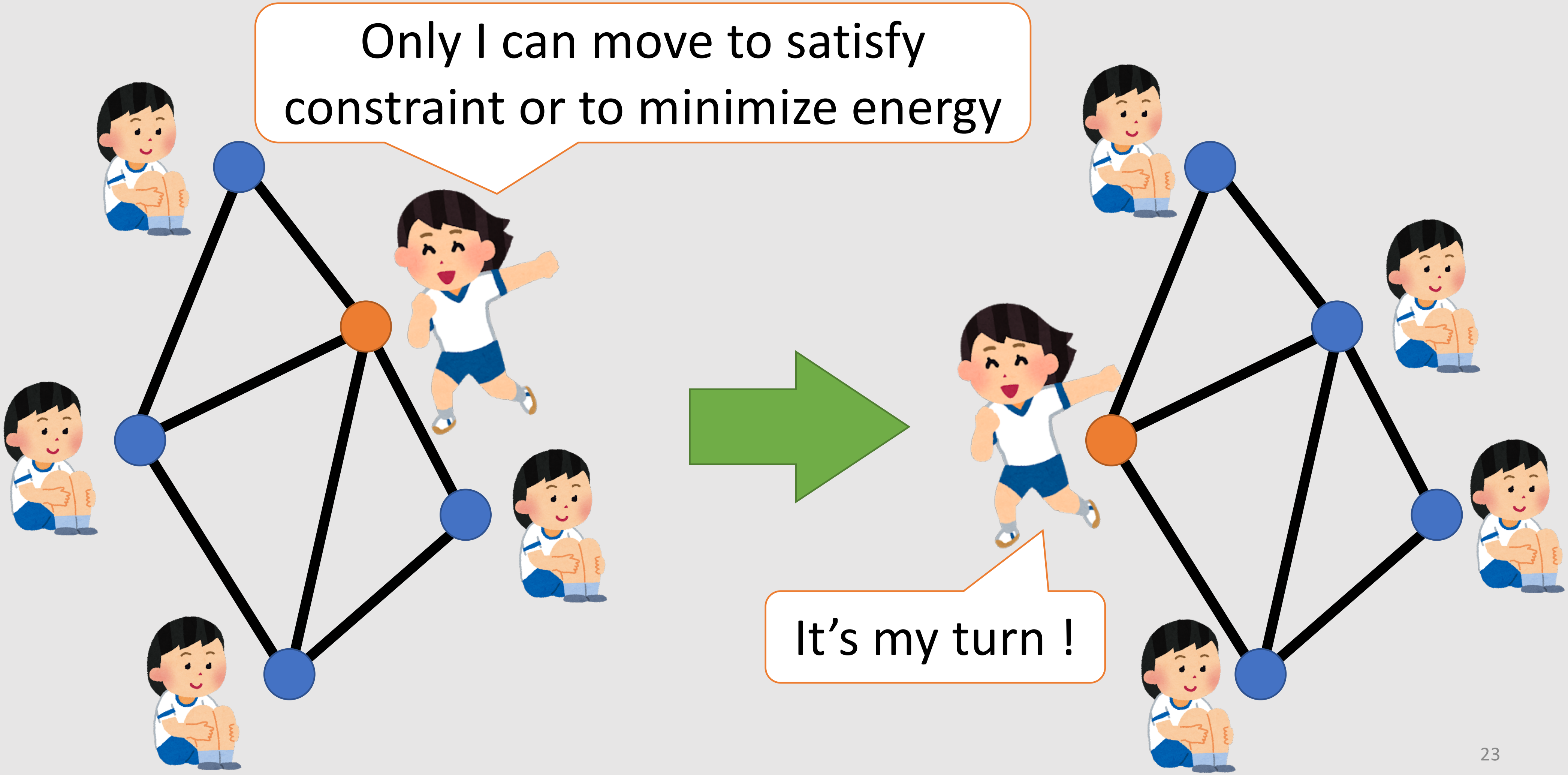
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$\Rightarrow x_1 = (b_1 - a_{12}x_2 - \cdots - a_{1n}x_n)/a_{11}$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$$\Rightarrow x_n = (b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots)/a_{nn}$$

Gauss-Seidel Method in a Grid



Gauss-Seidel Method in Matrix Form

$$(D + L + U)x = b$$

$$(D + L)x^k + Ux^{k-1} = b$$

$$x^k = (D + L)^{-1}\left(b - Ux^{k-1}\right)$$

Jacobi Method

1. Solve each row independently to obtain $\boldsymbol{x}'$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$\Rightarrow x_1' = (b_1 - a_{12}x_2 - \cdots - a_{1n}x_n)/a_{11}$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$$\Rightarrow x_n' = (b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots)/a_{nn}$$

2. Update solution at the same time as $\boldsymbol{x} = \boldsymbol{x}'$

Jacobi Method in Matrix Form

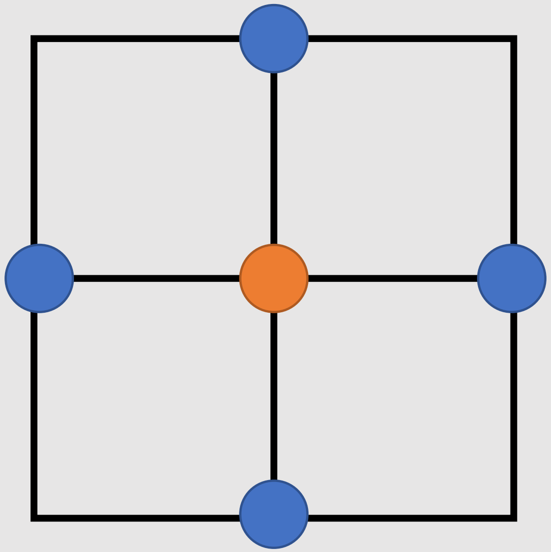
$$(D + L + U)x = b$$

$$Dx^k + (L + U)x^{k-1} = b$$

$$x^k = D^{-1}\{b - (L + U)x^{k-1}\}$$

Stencil of a 2D Regular Grid

- Stencil represents the diagonal & off-diagonal component of matrix for a row



graph Laplacian stencil

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & \boxed{4} & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

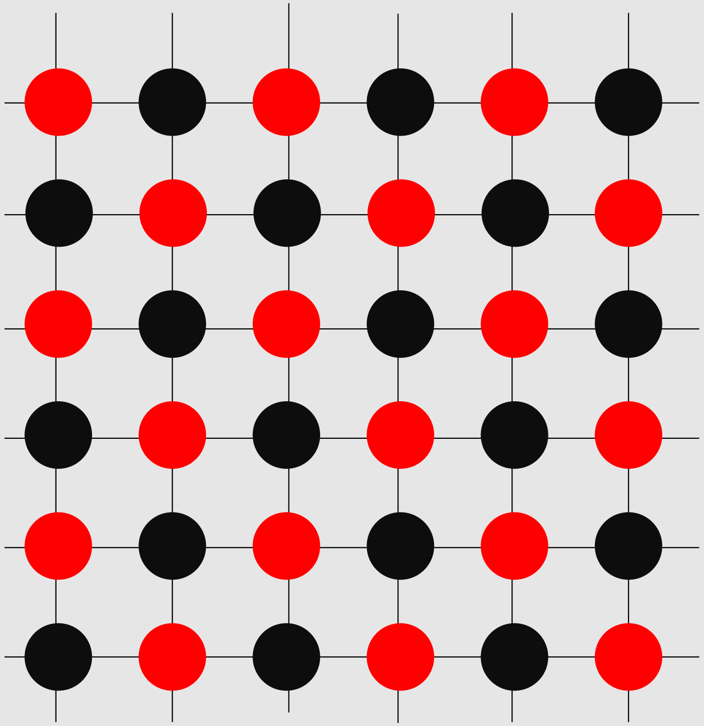
diagonal component

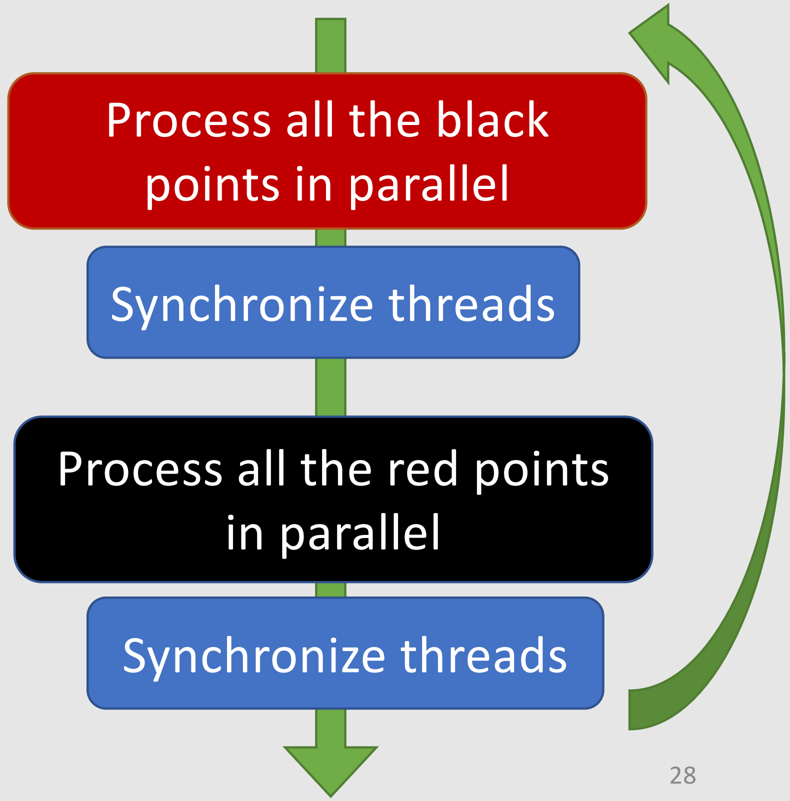
stencil in real life

credit: bukk @ wikipedia

Red-Black Ordering for Regular Grid

- The data of same color can be processed in any order (no-synchronization is necessary for parallel computation)

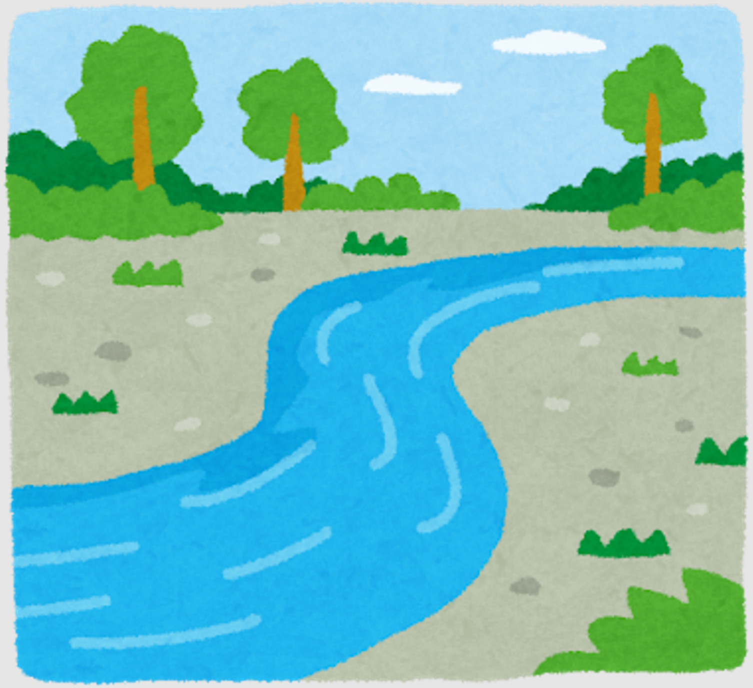




Lagrangian vs. Eulerian

Temperature of a River

- How to record the history of temperature of the flowing water?



Reference Frames

Lagrangian

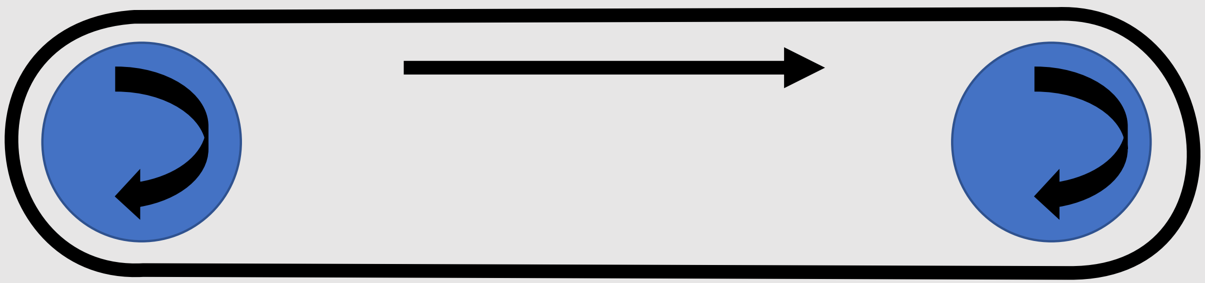
Observation point is moving together with flow

Eulerian

Observation point is fixed

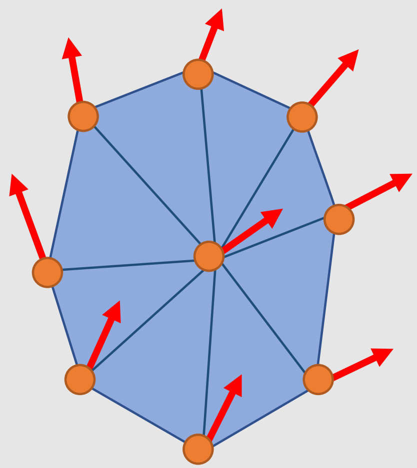
Material Derivative

- Measuring the change of the temperature on the carousel



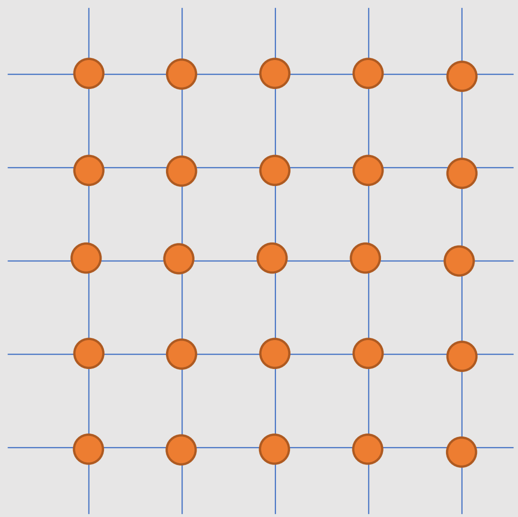
Data Structure for Continuum

Lagrangian
(e.g., deformable mesh)



Observation points moves over time

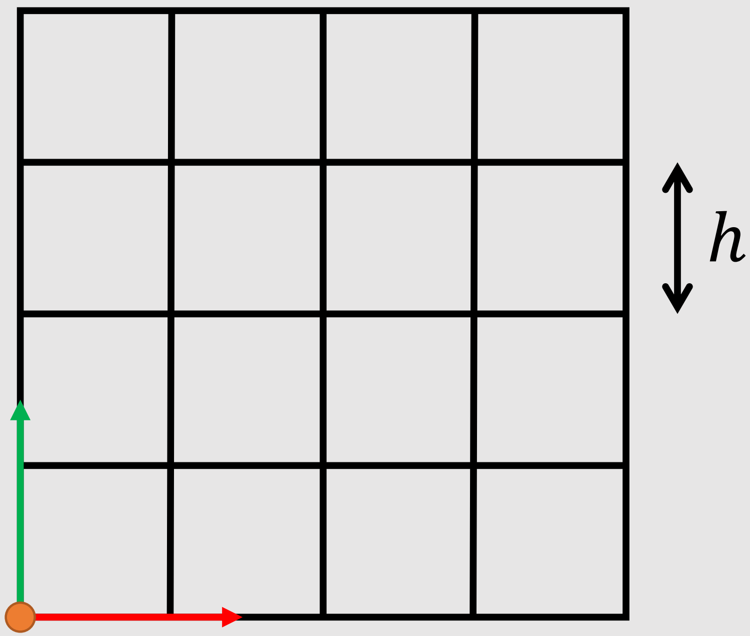
Eulerian
(e.g., regular grid)



Observation points don't move

Regular Grids

- Most common discretization for spatial values



Let's find out the corresponding grid cell for (p_x, p_y)

Check it out!

Lagrange Multiplier Method

ラグランジュ未定乗数法

Why Constraints?

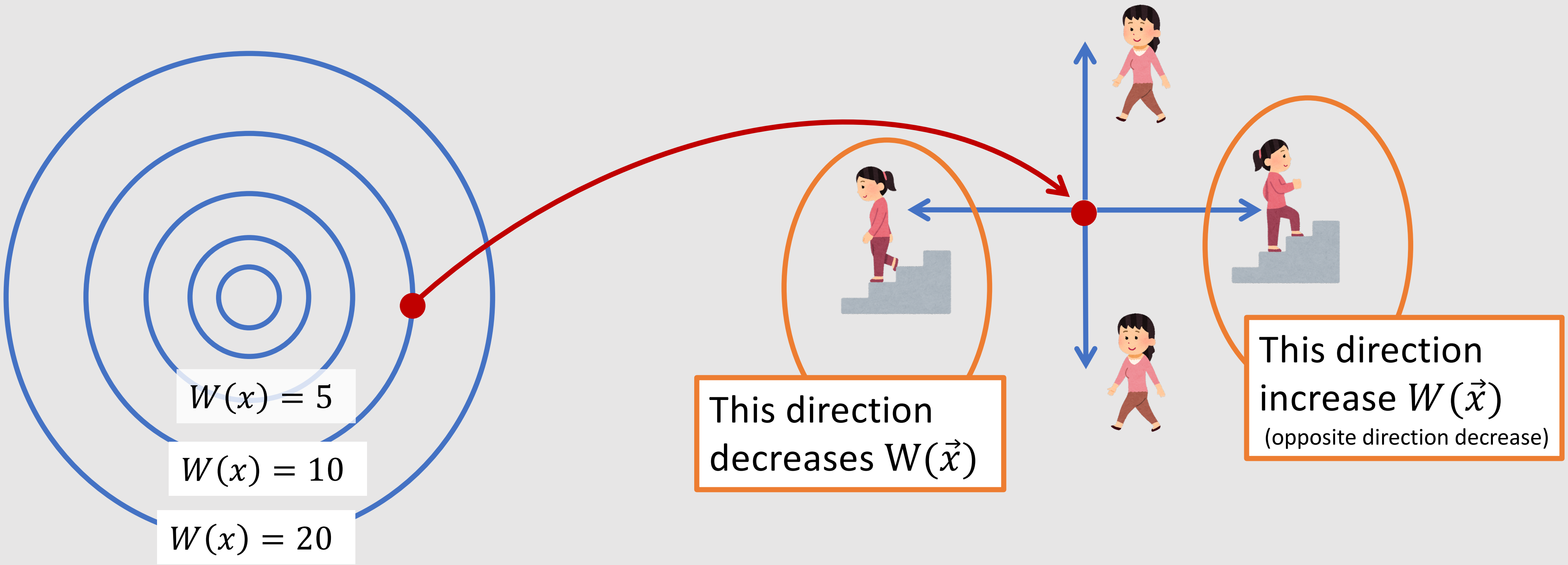
- Solid deformation
  - Non penetration constraints

Credit: Damnsoft 09 @ Wikipedia

- Fluid
  - incompressibility constraints: vortex

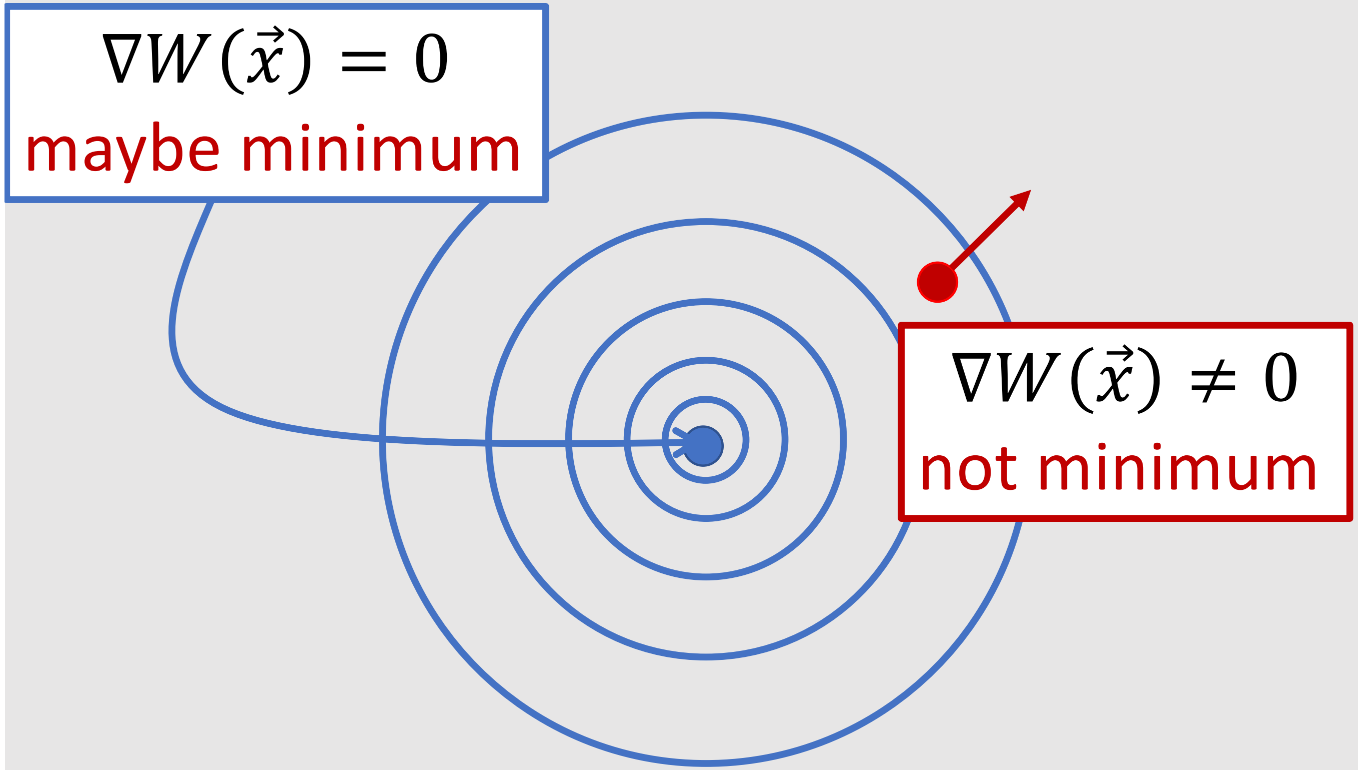
Credit: Astrobob @ Wikipedia

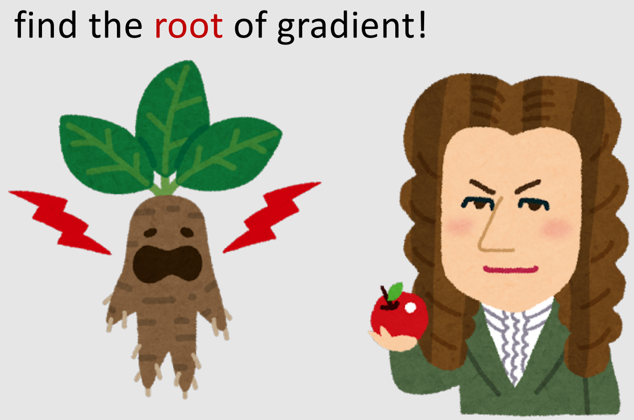
Not Minimum If Its Gradient is not Zero



Maybe Minimum if Gradient is Zero

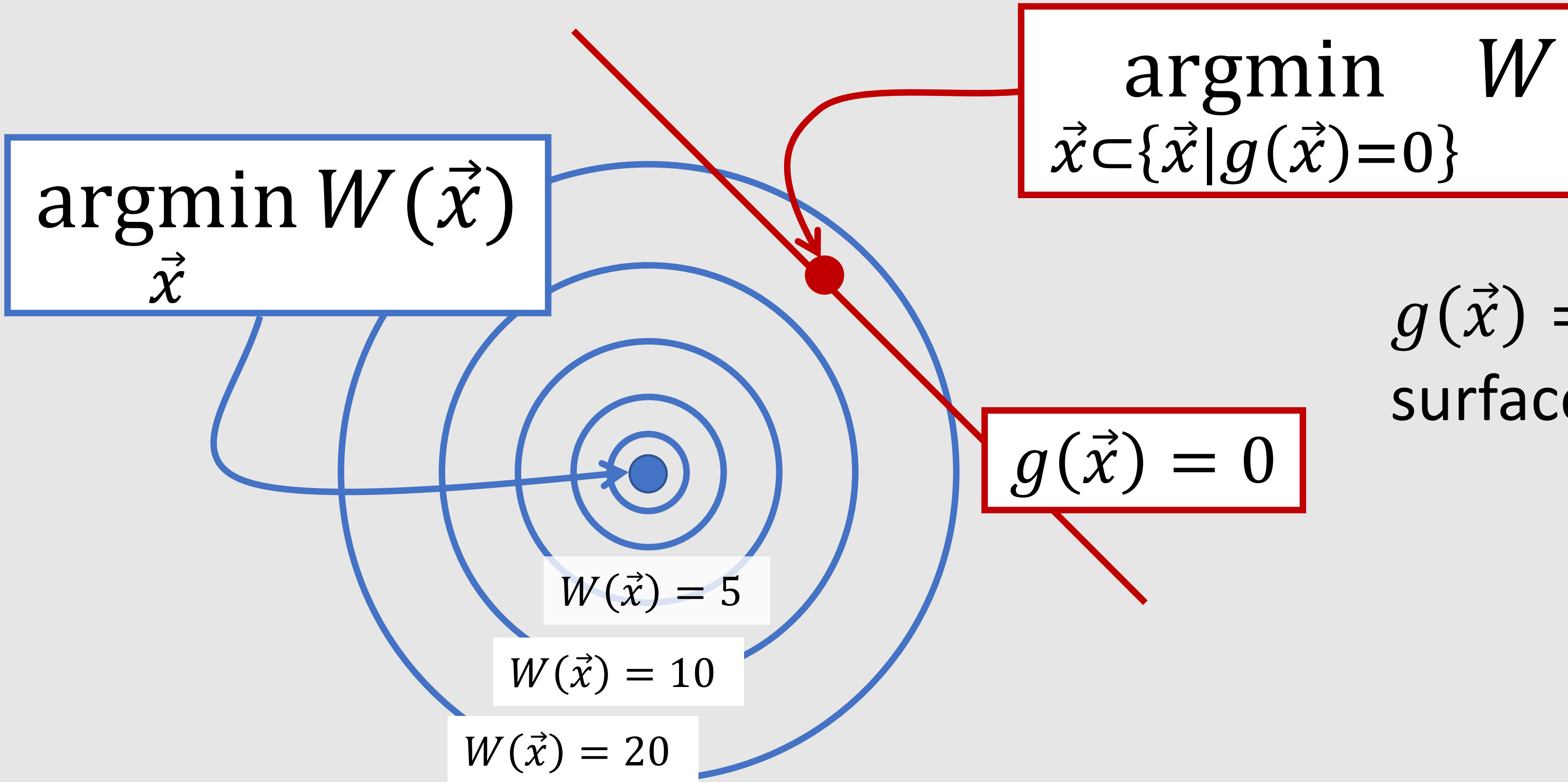
- Find a candidate where the gradient is zero $\nabla W(\vec{x}) = 0$





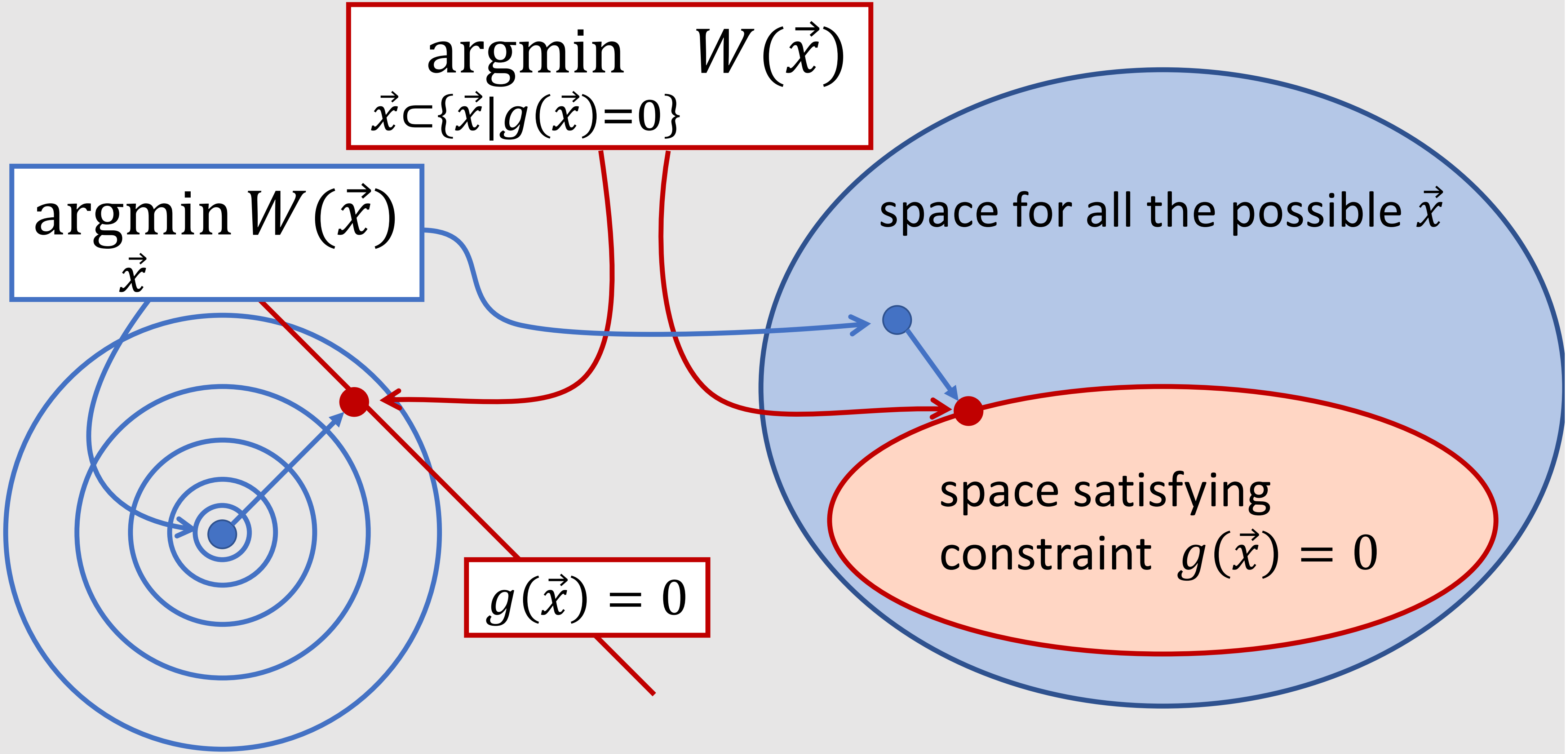
Optimization with Constraint

- Find a point $\vec{x}$ where the function $W(\vec{x})$ is minimized while satisfying $g(\vec{x}) = 0$



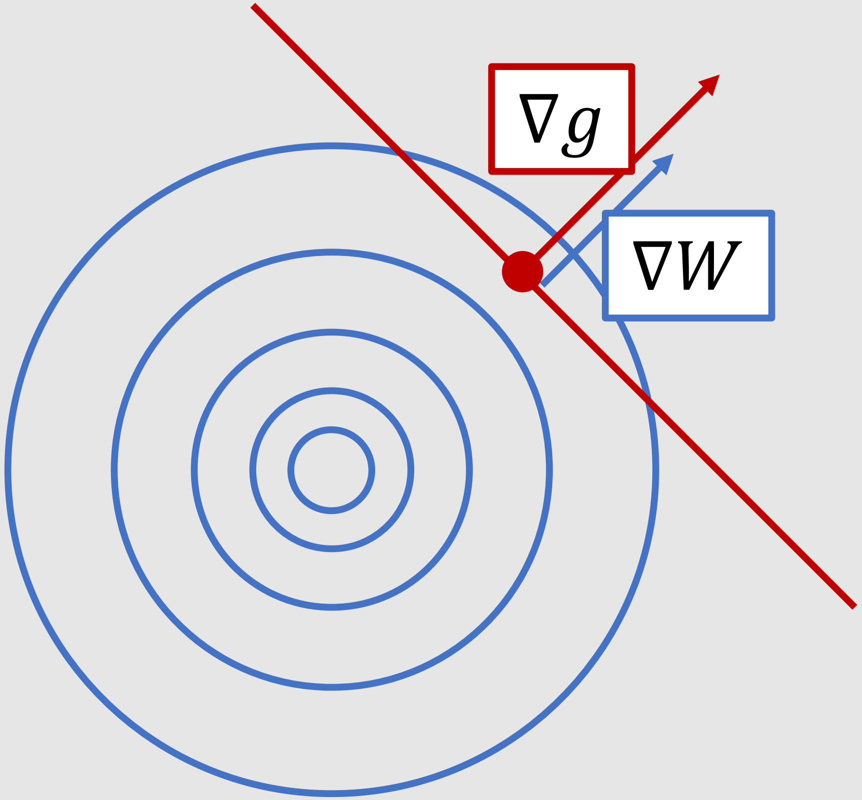
$g(\vec{x}) = 0$ is an implicit surface representation

Abstract View of the Solution Space



Lagrange Multiplier Method

- At minimum point, two gradients $\nabla W, \nabla g$ should be parallel



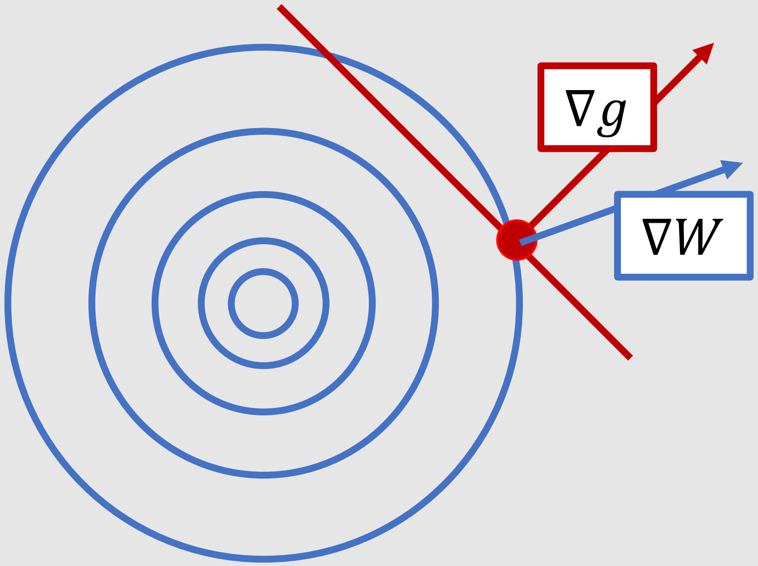
$$\nabla W \parallel \nabla g$$

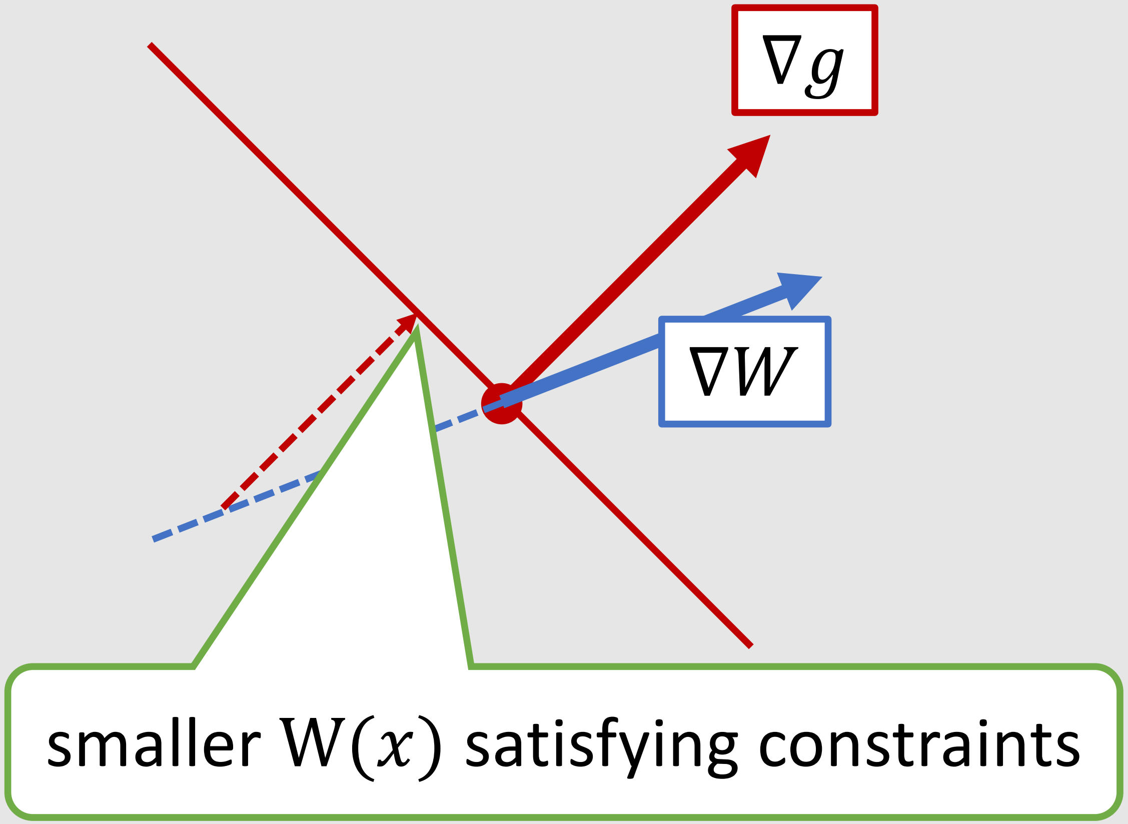
$$\Downarrow$$

$$\exists \lambda \neq 0 \;\; s.t. \; \nabla W = \lambda \nabla g$$

Why Parallel at Constrained Minimum?

- If $\nabla W, \nabla g$ are not parallel, smaller $W(x)$ exists satisfying constraints





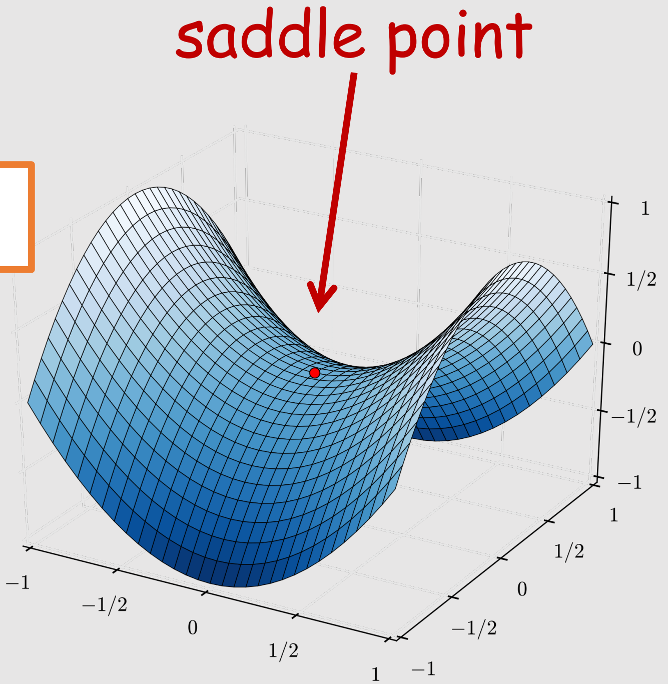
Find Saddle Point not Minima for LM Method

- We changed minimization problem to saddle point finding problem

$$\nabla W(\vec{x}) = \lambda \nabla g(\vec{x})$$

$$\nabla \overline{W}(\vec{x}, \lambda) = 0 \; where \; \overline{W}(\vec{x}, \lambda) = W(\vec{x}) - \lambda g(\vec{x})$$

Don’t minimize $\overline{W}\ (\vec{x}, \lambda)$. Find where the gradient is zero using the Newton method



Credit: Nicoguaro @ Wikipedia

Lin. System for Lagrange Multiplier Method

$$\begin{pmatrix} \nabla W(\vec{x}) - \lambda \nabla g(\vec{x}) \\ -g(\vec{x}) \end{pmatrix} = H(\vec{x}, \lambda) = 0$$

Newton-Raphson method

$$\begin{pmatrix} d\vec{x} \\ d\lambda \end{pmatrix} = -[\nabla H]^{-1} H$$

$$= -\begin{bmatrix} \nabla^2 W(\vec{x}) - \lambda \nabla^2 g(\vec{x}) & -\nabla g(\vec{x}) \\ -\nabla g(\vec{x}) & 0 \end{bmatrix} \begin{pmatrix} \nabla W(\vec{x}) - \lambda \nabla g(\vec{x}) \\ -g(\vec{x}) \end{pmatrix}$$

Let's Practice Lagrange Multiplier Method

Maximize $f(x, y) = x + y$ where $g(x, y) = x^2 + y^2 - 1 = 0$

check it out!