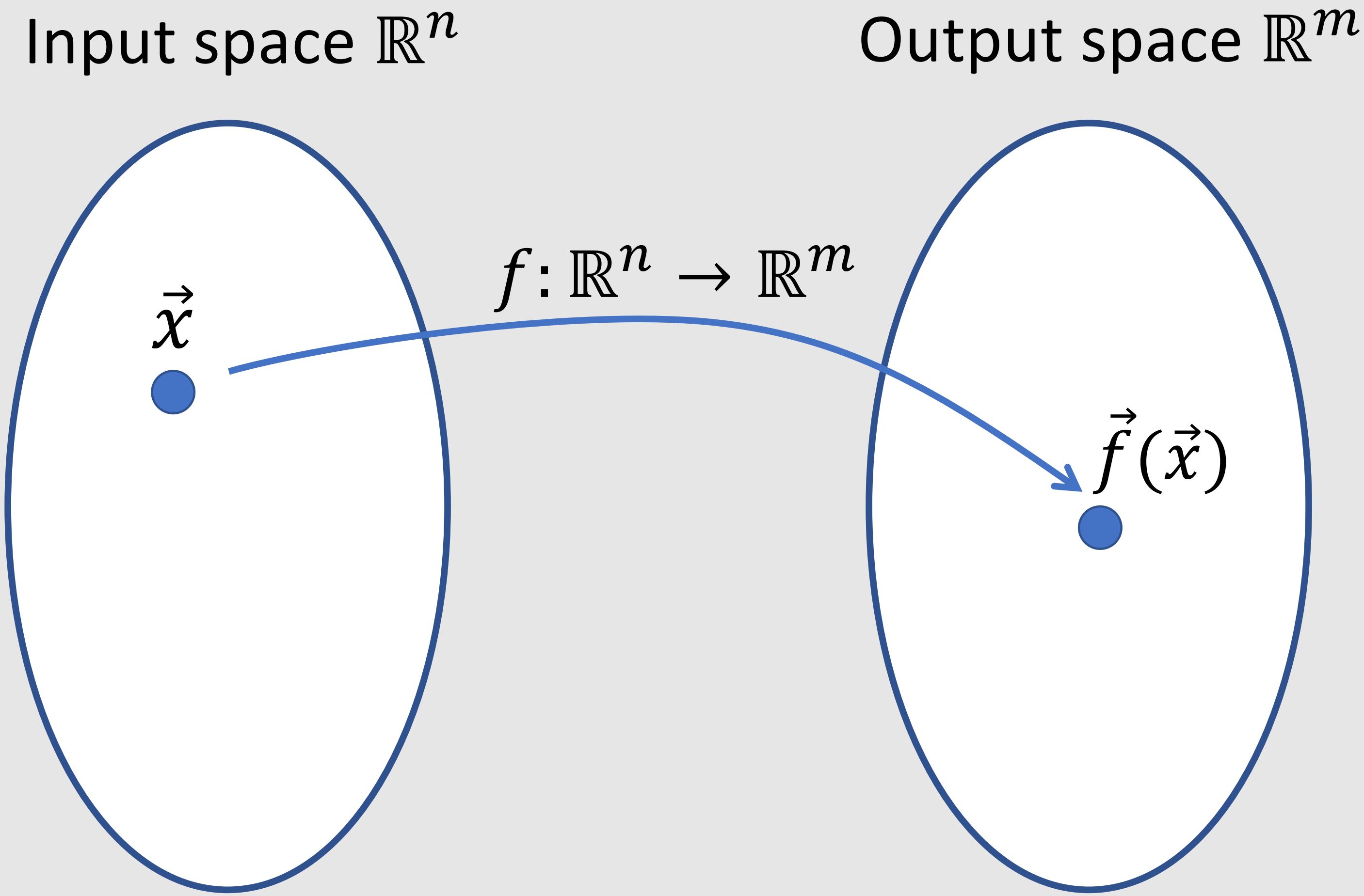
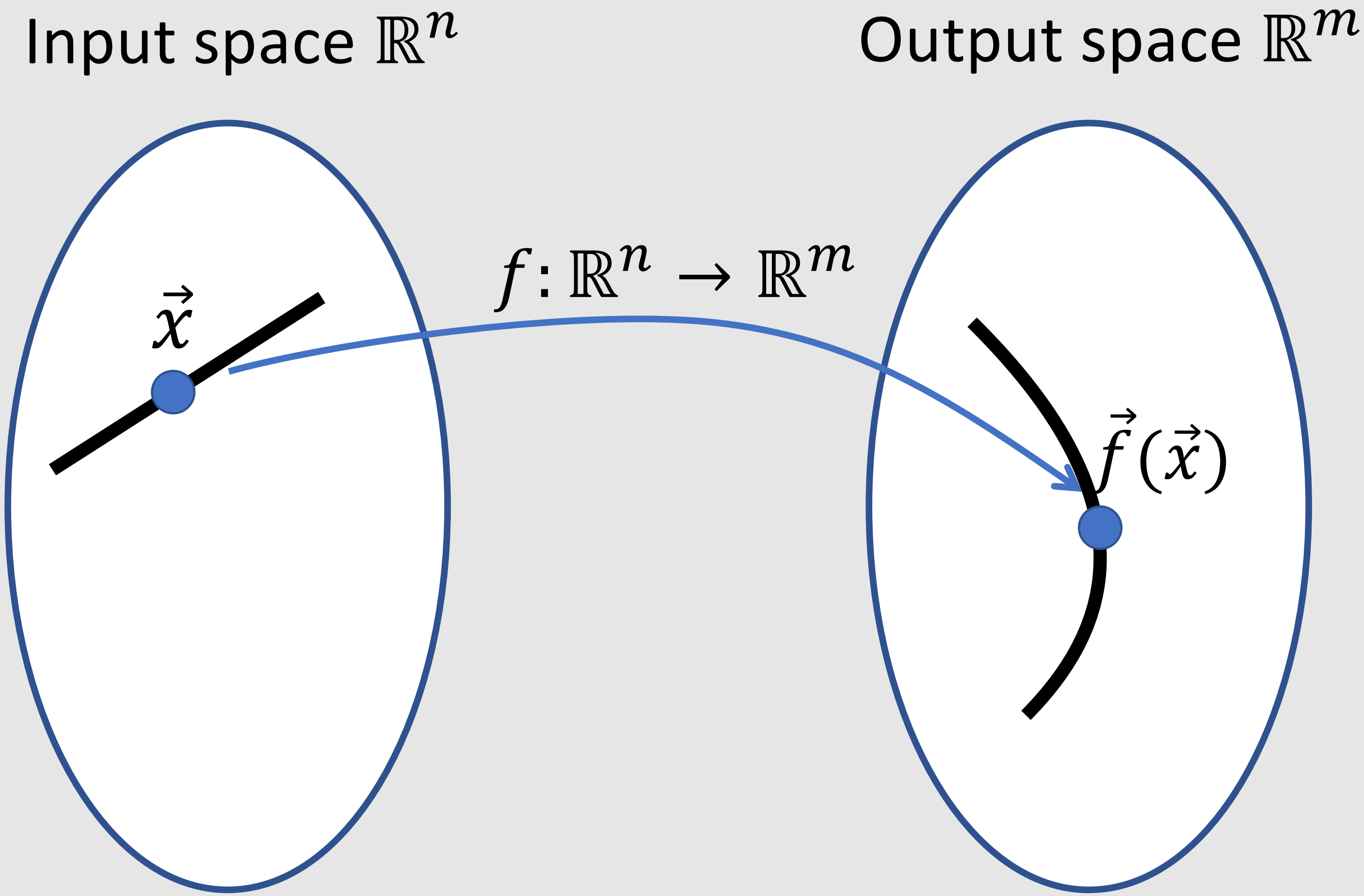
Jacobian & Hessian

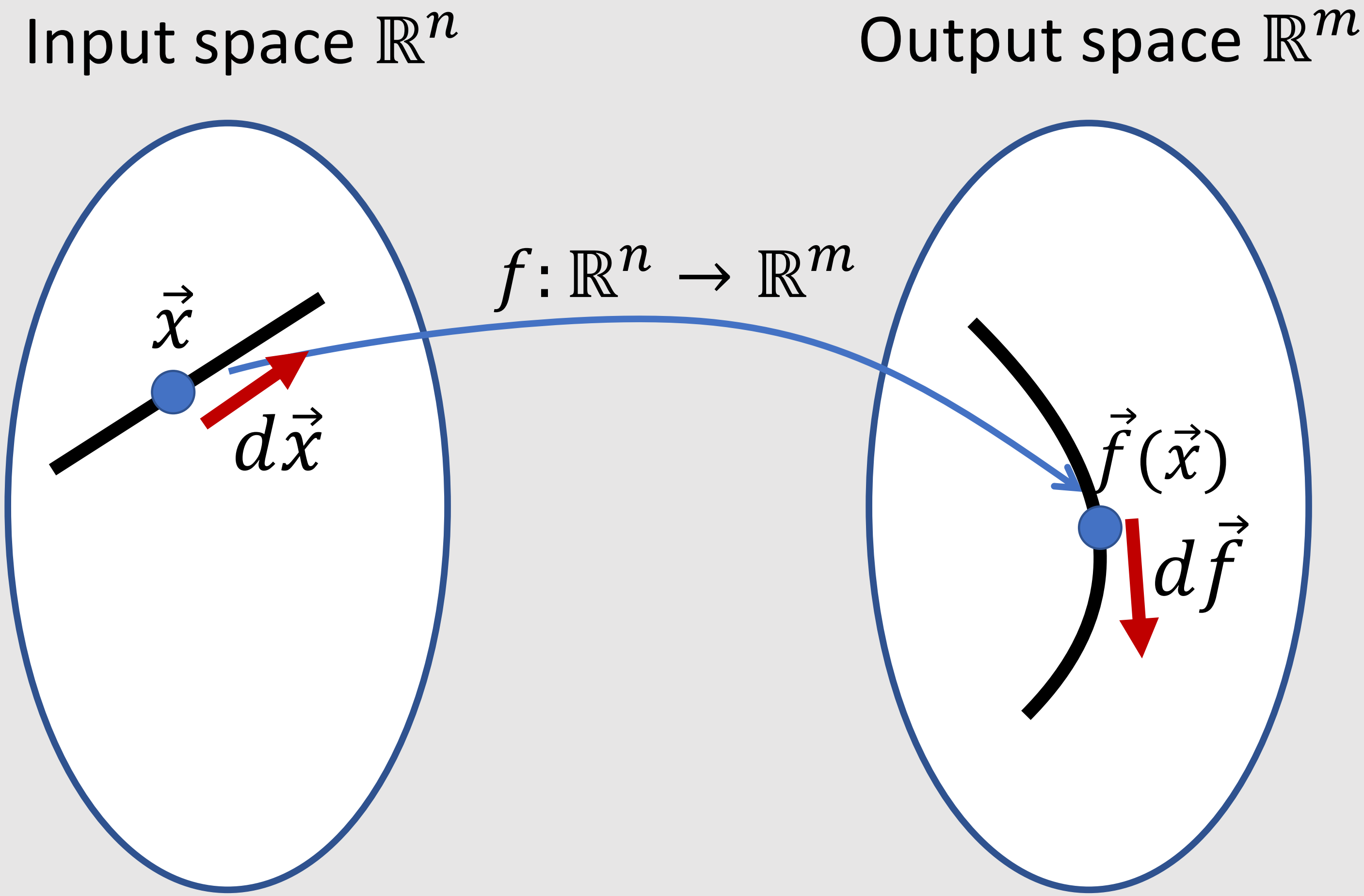
Multivariate Function: High Dimensional Map



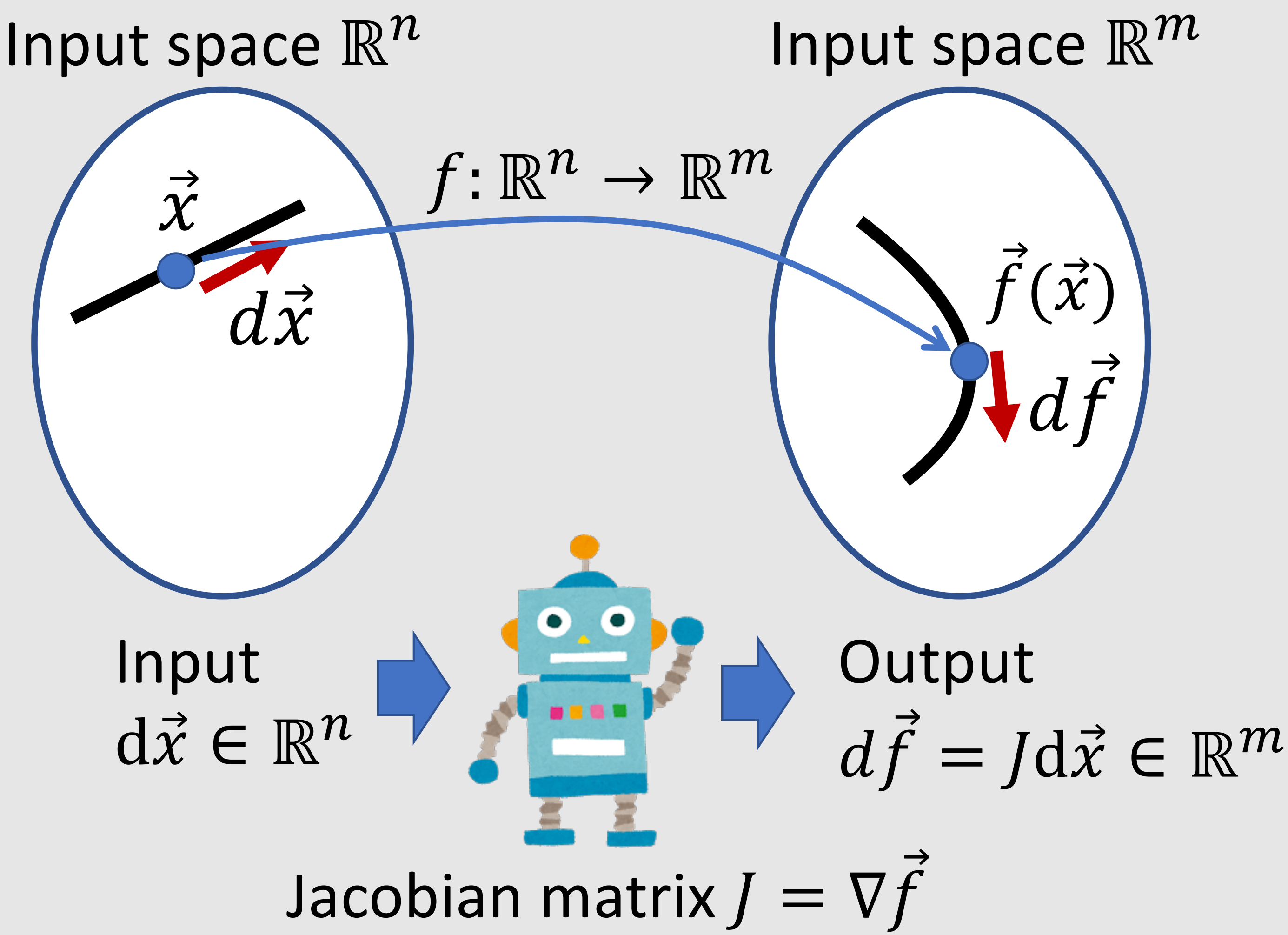
Trajectory of the Function



Differentiation of the Map

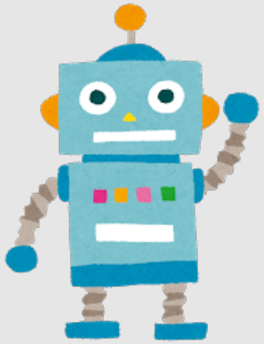


Jacobian Matrix: Gradient of Map



Jacobian Matrix: Gradient of Map

Input
$\mathrm{d}\vec{x} \in \mathbb{R}^n$

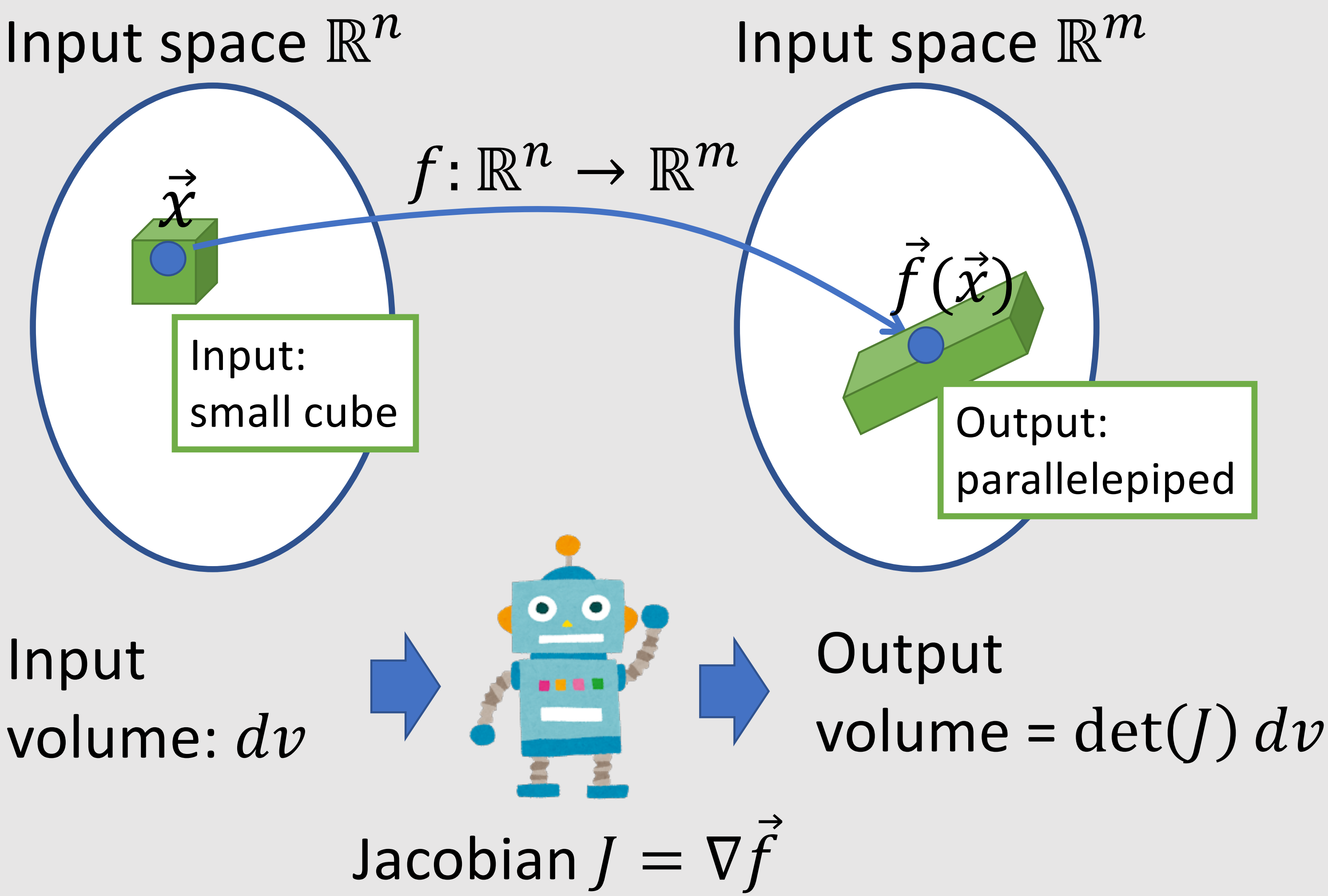


Output
$d\vec{f} = J\mathrm{d}\vec{x} \in \mathbb{R}^m$

Jacobian matrix $J = \nabla\vec{f}$

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial \mathbf{f}}{\partial x_1} & \cdots & \dfrac{\partial \mathbf{f}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \nabla^{\mathrm{T}} f_1 \\ \vdots \\ \nabla^{\mathrm{T}} f_m \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

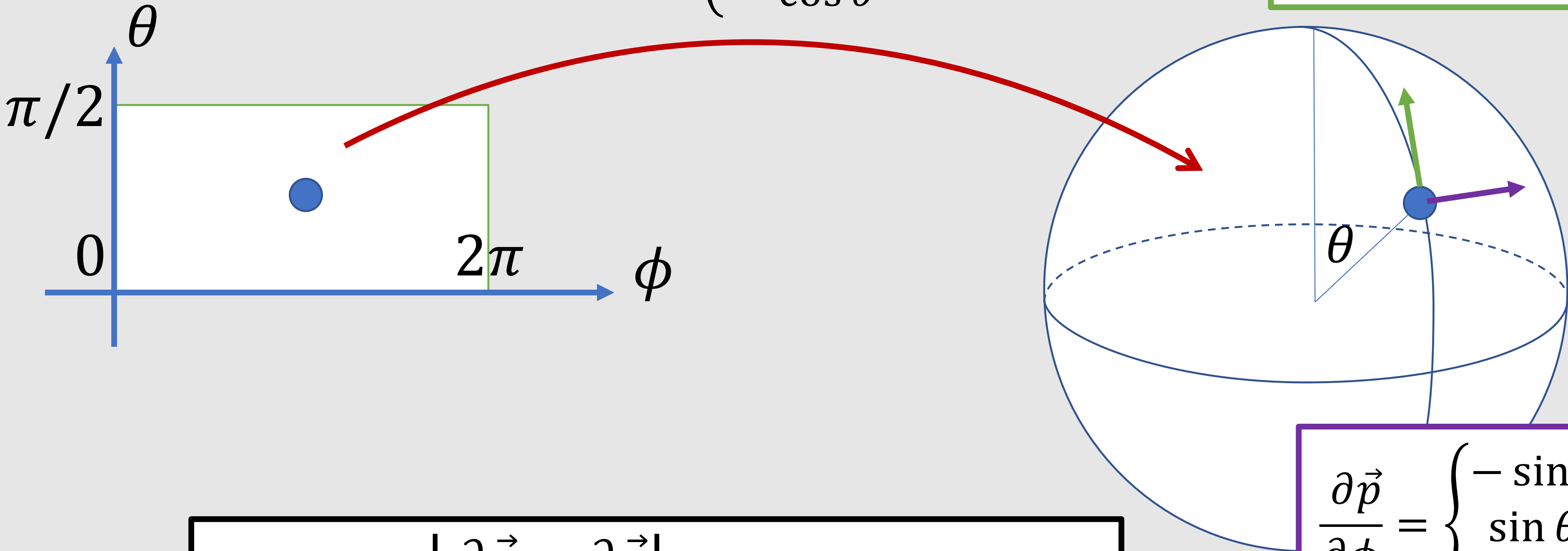
Jacobian Determinant: Volume Change Ratio



Spherical Coordinate

$$\vec{p} = \begin{cases} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{cases}$$

$$\frac{\partial\vec{p}}{\partial\theta} = \begin{cases} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{cases}$$



$$\frac{\partial\vec{p}}{\partial\phi} = \begin{cases} -\sin\theta\sin\phi \\ \sin\theta\cos\phi \\ 0 \end{cases}$$

$$\det(J) = \left|\frac{\partial\vec{p}}{\partial\theta} \times \frac{\partial\vec{p}}{\partial\phi}\right| = |\sin\theta\,\vec{p}| = \sin\theta$$

Hessian Matrix: Jacobian Matrix for Gradient

- Second derivative of a scalar function $f(\vec{x})$

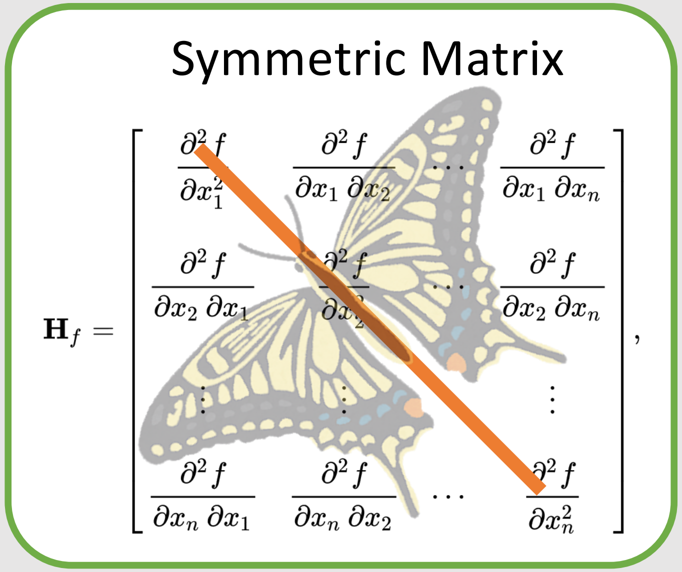
$$\boldsymbol{H}_f = \boldsymbol{J}(\nabla f(\vec{x}))$$

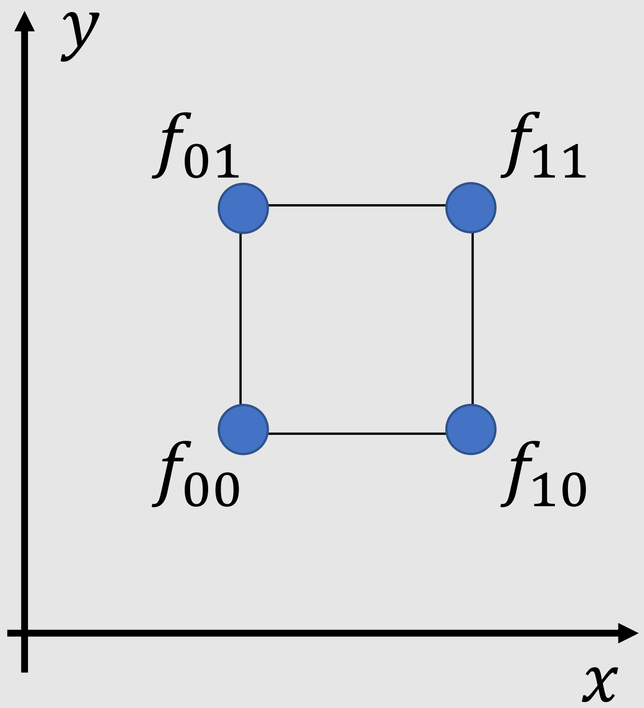
$$(\mathbf{H}_f)_{i,j} = \frac{\partial^2 f}{\partial x_i\, \partial x_j}.$$

$$\mathbf{H}_f = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1\, \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1\, \partial x_n} \\[2.2ex] \dfrac{\partial^2 f}{\partial x_2\, \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2\, \partial x_n} \\[2.2ex] \vdots & \vdots & \ddots & \vdots \\[2.2ex] \dfrac{\partial^2 f}{\partial x_n\, \partial x_1} & \dfrac{\partial^2 f}{\partial x_n\, \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix},$$

Symmetricity of Hessian

- Hessian is symmetric if $f(\vec{x})$ is continuous





$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \approx (f_{11} - f_{10}) - (f_{01} - f_{00})$$

equal

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \approx (f_{11} - f_{01}) - (f_{10} - f_{00})$$