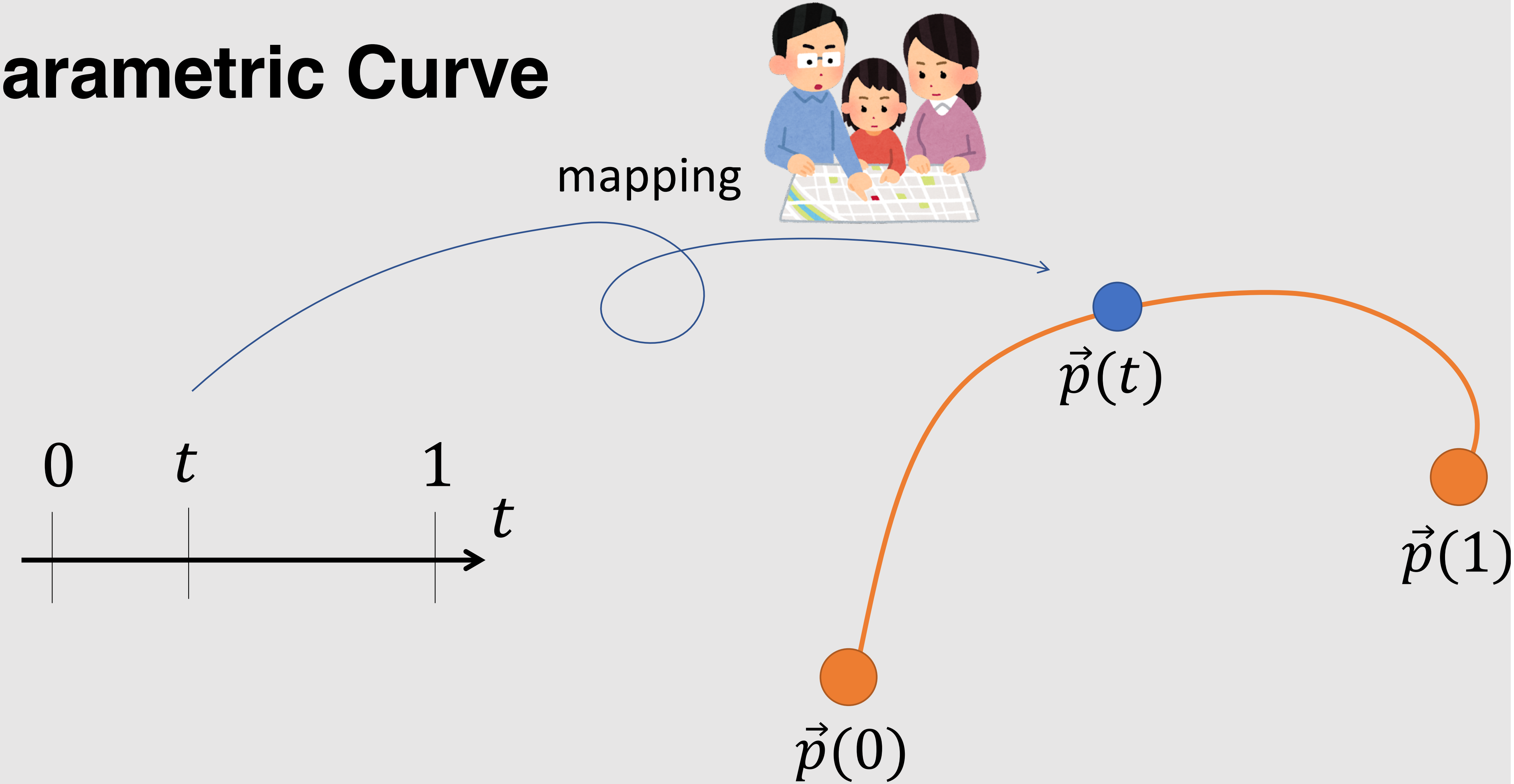
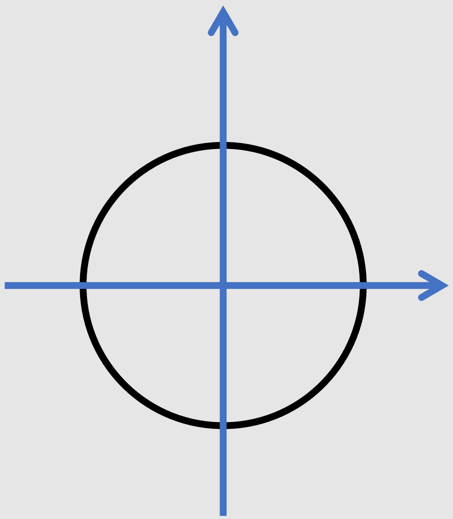
Parametric Curve

Parametric Curve



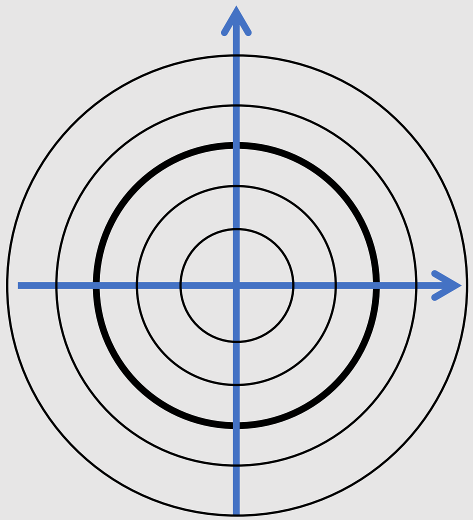
Example of Parametric Representation

$$|x^2 + y^2| = 1$$



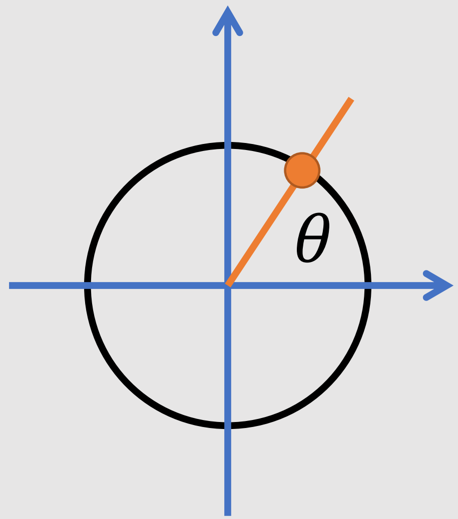
Implicit Representation

$$f(x, y) = |x^2 + y^2| - 1$$

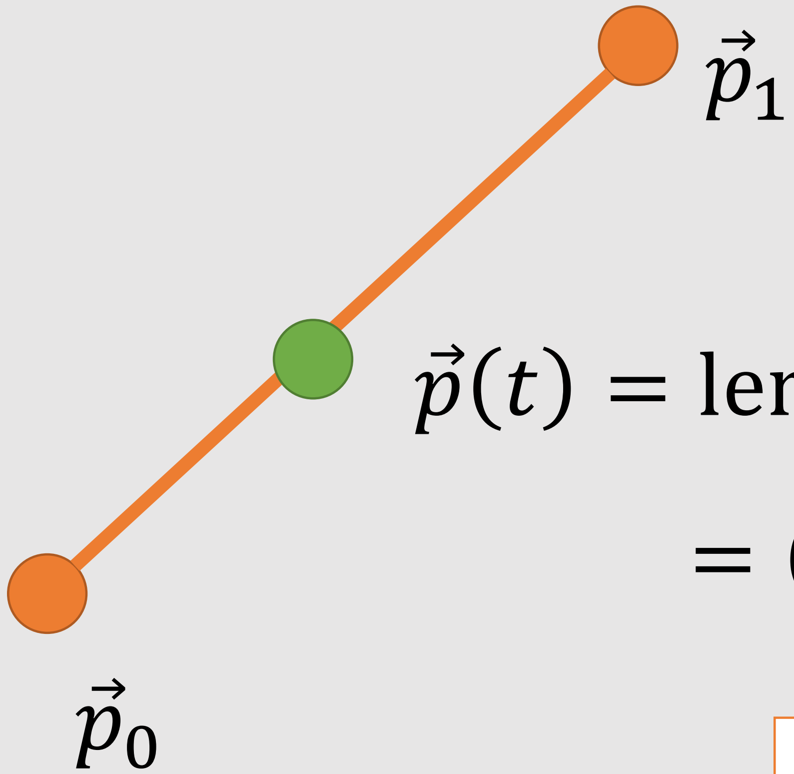


Parametric Representation

$$\begin{cases} x = \cos(2\pi\theta) \\ y = \sin(2\pi\theta) \end{cases}$$



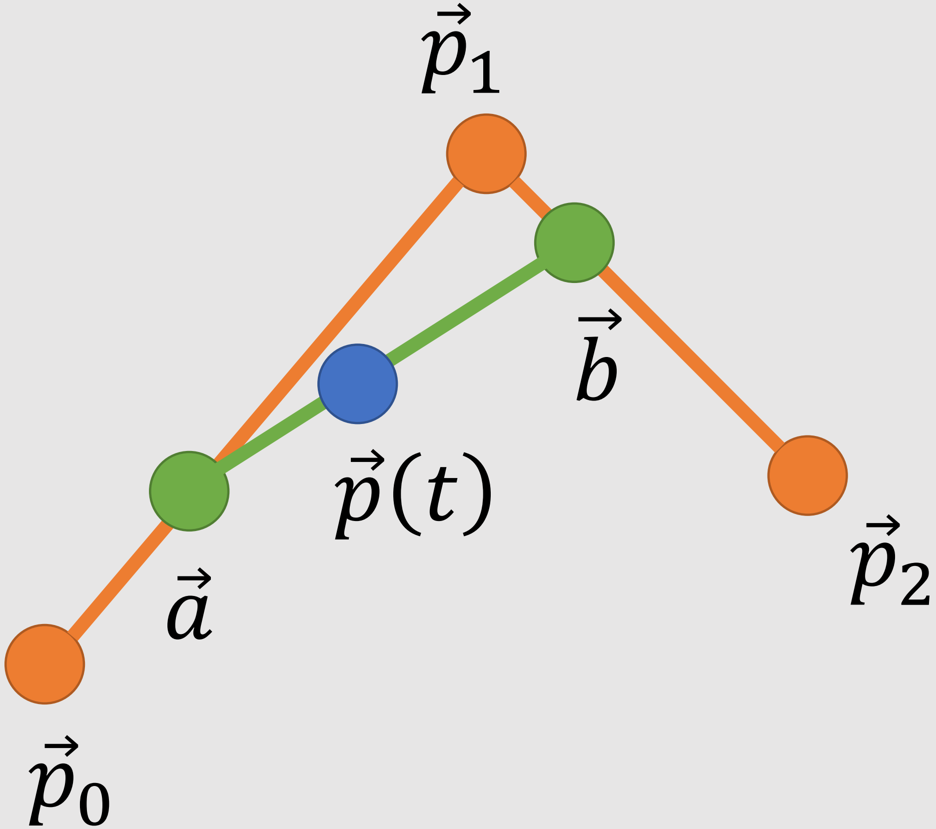
Simplest Paramtric Curve: Line



$$\vec{p}(t) = \text{lerp}(\vec{p}_0, \vec{p}_1, t)$$
$$= (1 - t)\vec{p}_0 + t\vec{p}_1$$

lerp=Linear Interpolation

Quadratic Bézier Curve

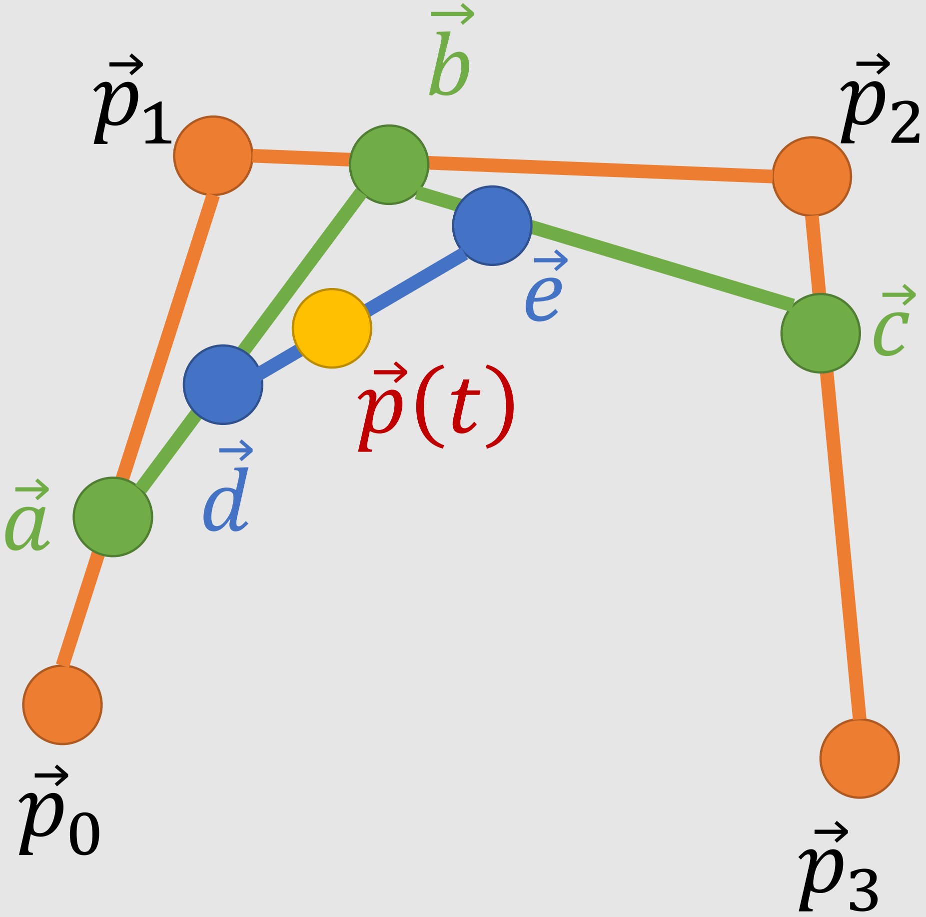


$$\vec{a}(t) = \mathrm{lerp}(\vec{p}_0, \vec{p}_1, t)$$

$$\vec{b}(t) = \mathrm{lerp}(\vec{p}_1, \vec{p}_2, t)$$

$$\vec{p}(t) = \mathrm{lerp}(\vec{a}, \vec{b}, t)$$

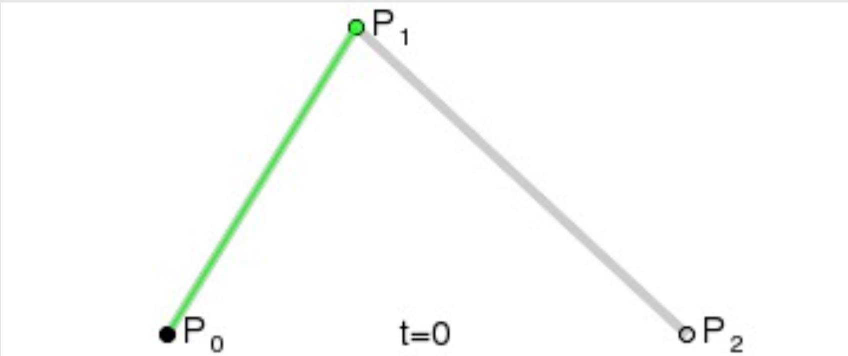
Cubic Bézier Curve



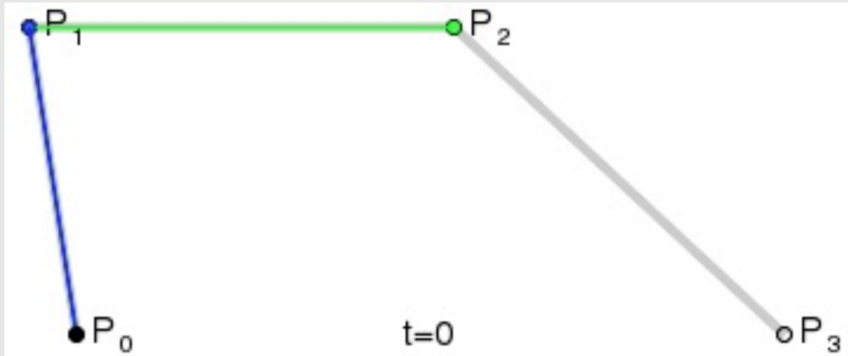
$$\vec{a}(t) = \mathrm{lerp}(\vec{p}_0, \vec{p}_1, t)$$
$$\vec{b}(t) = \mathrm{lerp}(\vec{p}_1, \vec{p}_2, t)$$
$$\vec{c}(t) = \mathrm{lerp}(\vec{p}_2, \vec{p}_3, t)$$
$$\vec{d}(t) = \mathrm{lerp}(\vec{a}, \vec{b}, t)$$
$$\vec{e}(t) = \mathrm{lerp}(\vec{b}, \vec{c}, t)$$
$$\vec{p}(t) = \mathrm{lerp}(\vec{d}, \vec{e}, t)$$

De Casteljau's Algorithm

Quadratic Bézier Curve



Cubic Bézier Curve



Bernstein Polynomial Form

$$\vec{a}(t) = (1 - \mathrm{t})\vec{\mathrm{p}}_0 + t\vec{p}_1$$

$$\vec{b}(t) = (1 - t)\vec{p}_1 + t\vec{p}_2$$

$$\vec{c}(t) = (1 - t)\vec{p}_2 + t\vec{p}_3$$

$$\vec{d}(t) = (1 - t)\vec{a} + t\,\vec{b}$$

$$\vec{e}(t) = (1 - t)\vec{b} + t\,\vec{c}$$

$$\vec{p}(t) = (1 - t)\vec{d} + t\vec{e}$$

$$\vec{p}(t) = \vec{p}_0\{(1-t)^3\} + \vec{p}_1\{3t(1-t)^2\} + \vec{p}_2\{3(1-t)^2 t\} + \vec{p}_3\{t^3\}$$

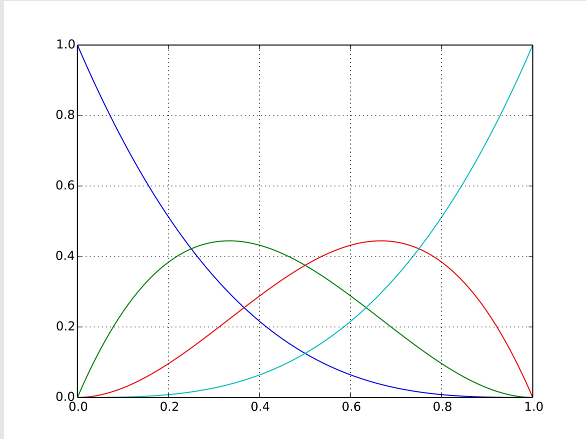
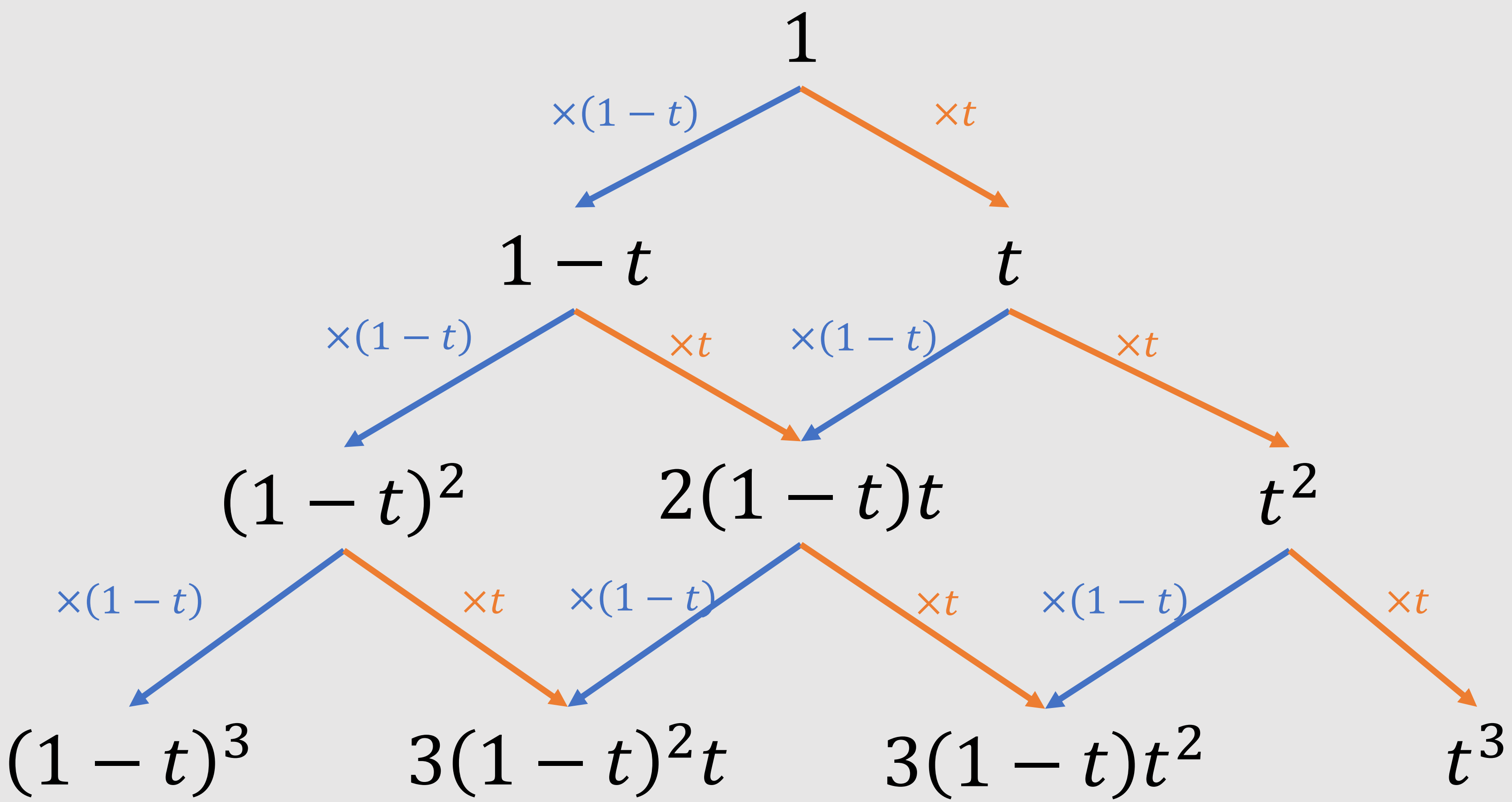


Image Credit: Ben FrantzDale@Wikipedia

Bernstein's Polynomial



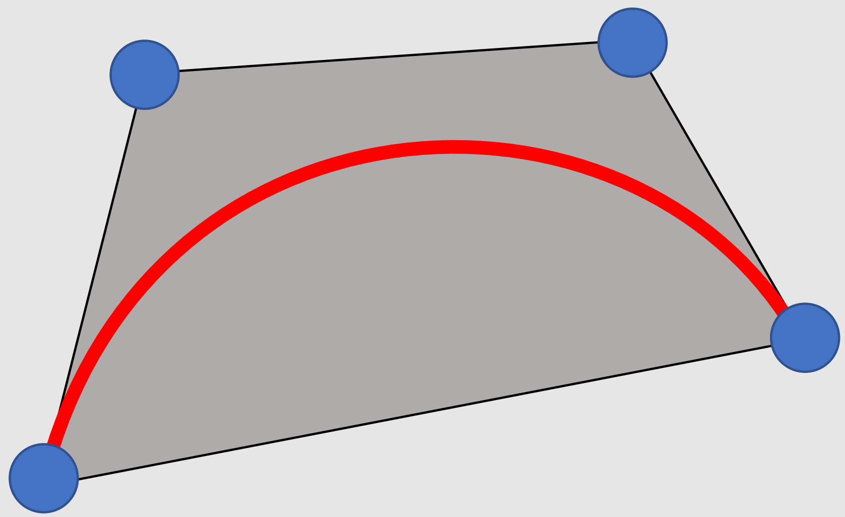
Bezier Curve & Convex Hull of Ctrl. Points

$$p(t) = \sum w_i(t)\vec{p}_i$$

$$\sum w_i(t) = 1$$

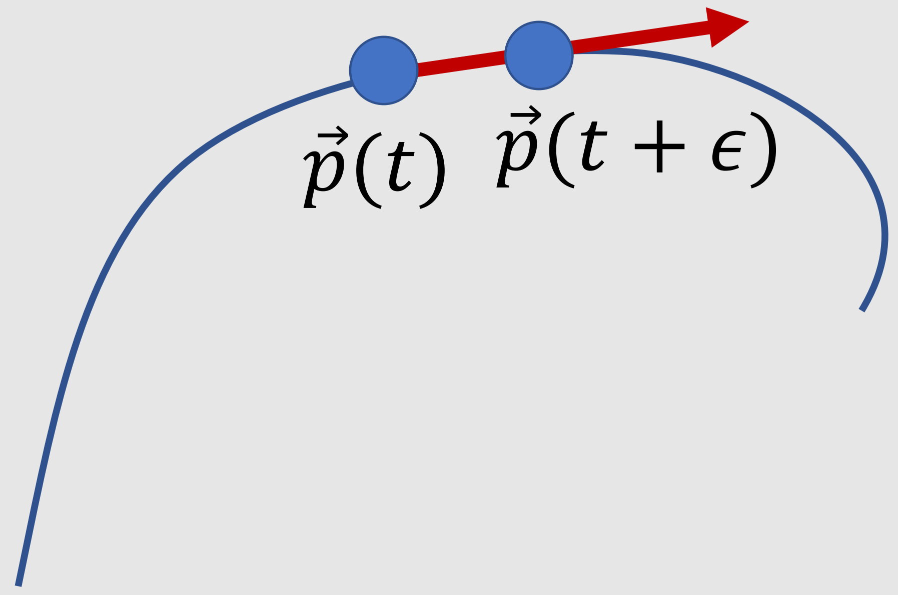
$$0 \leq w_i(t) \leq 1$$

Bézier curve is always inside the convex hull



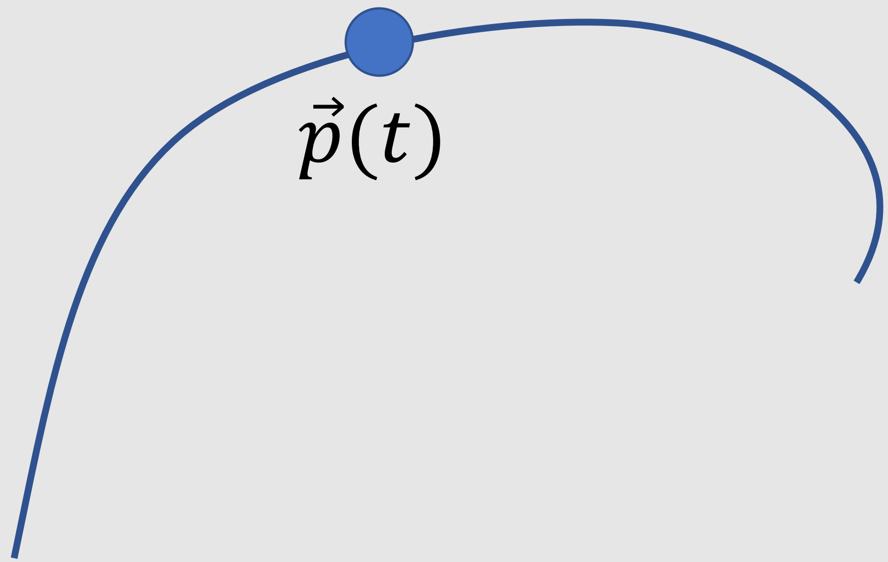
Derivative of a Parametric Curve = Tangent

$$\vec{p}(t)' = \lim_{\epsilon \to 0} \frac{\vec{p}(t+\epsilon) - \vec{p}(t)}{\epsilon}$$



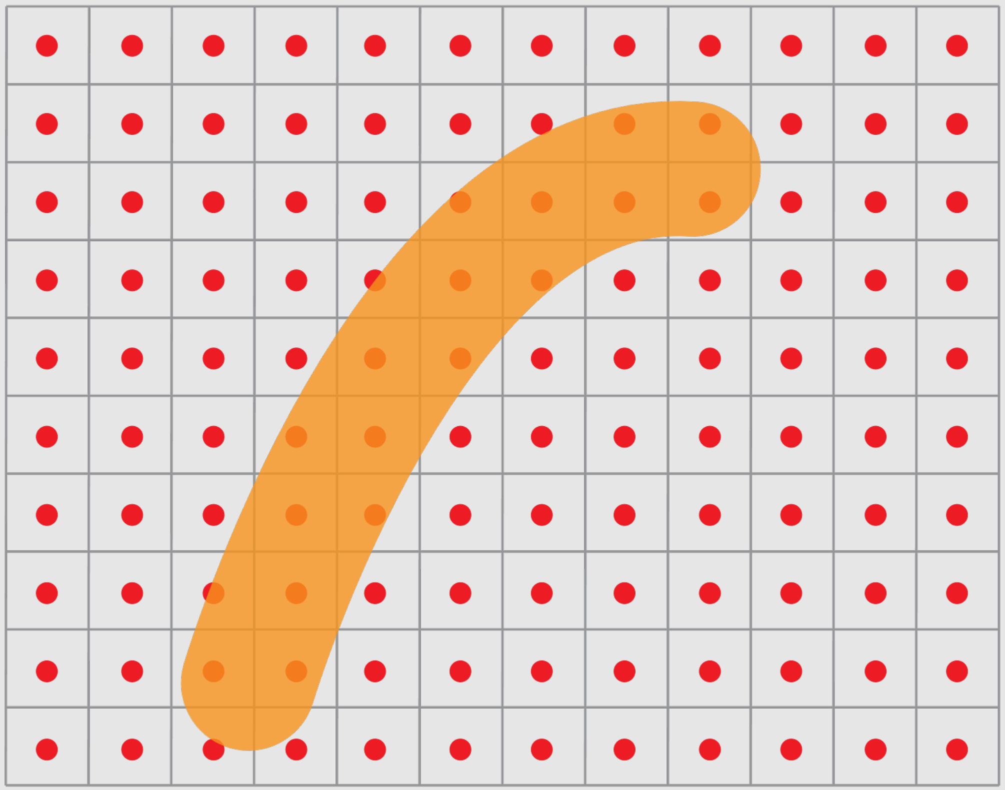
Length of a Parametric Curve

$$Length = \int_0^1 |\vec{p}(t)'|\, dt$$



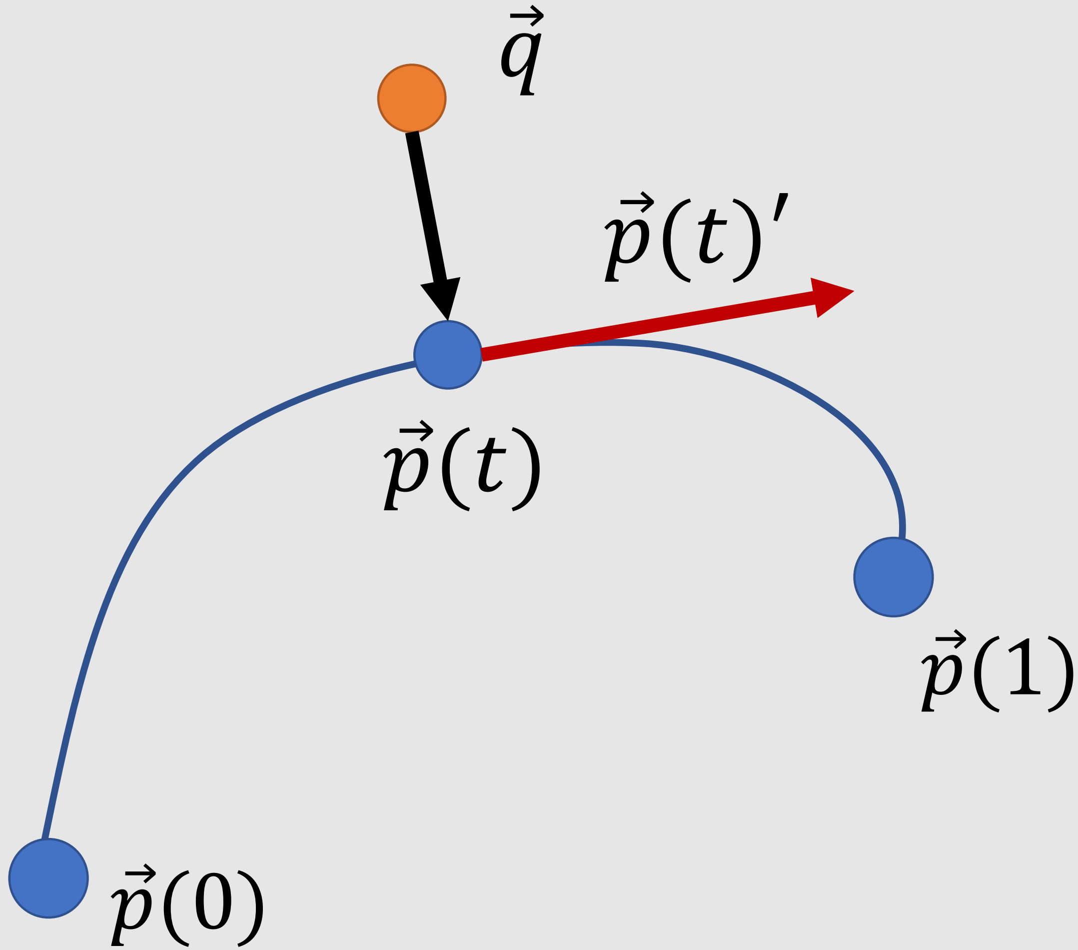
Rasterization of Curve

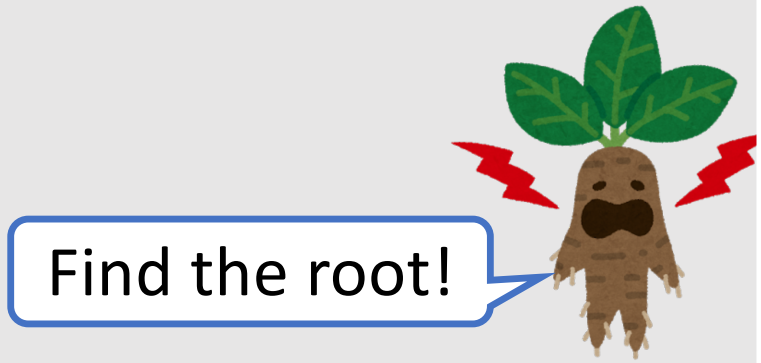
- Need to compute distance to the curve from each pixel



Distance to the Curve

$$Distance = \min_{t \in [0,1]} \|\vec{q} - \vec{p}(t)\|^2$$





Sturm's Method

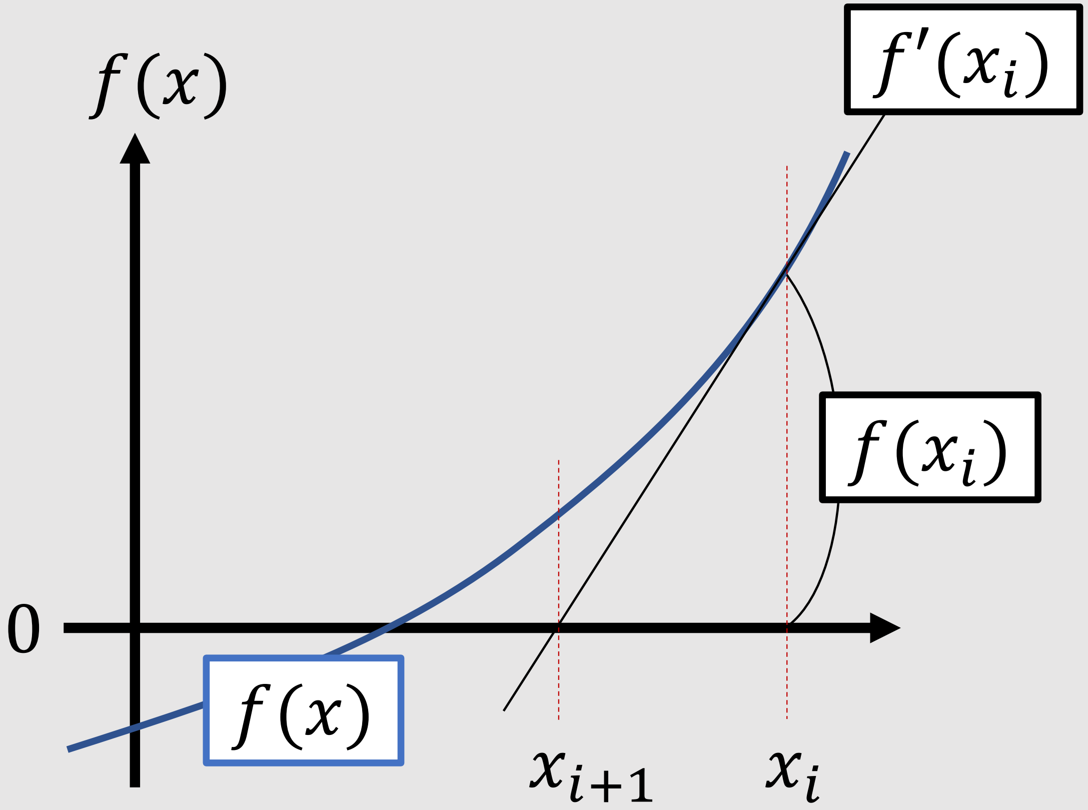
Horner’s Method

- Fast evaluation of polynominal: n multiplications and n additions

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

$$= a_0 + x\Big(a_1 + x\Big(a_2 + x\big(a_3 + \cdots + x(a_{n-1} + x a_n)\big)\Big)\Big)$$

Newton's Method to Find the Root



To find x where $f(x) = 0$

Iterate:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Division of Polynomial

divisor

5 ← quotient

5) 27 ← quotient

25

2 ← remainder

$Q(x)$

$g(x)$) $f(x)$

$R(x)$

$$27 = 5\times 5 + 2$$

$$\deg(2) < \deg(5)$$

$$f(x) = Q(x)g(x) + R(x)$$

$$\deg(R(x)) < \deg\big(g(x)\big)$$

Example of Polynomial Division

$$\begin{array}{r|l} & \mathrm{Q}(x) \\ g(x) & \overline{f(x)} \\ & \underline{} \\ & R(x) \end{array}$$

$$f(x) = ax^2 + bx + c$$

$$g(x) = dx + e$$

$$f(x) = Q(x)g(x) + R(x)$$

$$\deg(R(x)) < \deg\big(g(x)\big)$$

Sturm Sequence & Sturm's Method

Sturm sequence

$$f_0 = f$$

$$f_1 = f_0'$$

$$f_2 = -\mathrm{rem}(f_0, f_1)$$

$$\vdots$$

$$f_{i+1} = -\mathrm{rem}(f_{i-1}, f_i)$$

$$\vdots$$

Sturm Number → number of sign change: $N(x)$

x	$f_0(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	number of sign change: $N(x)$
a	$-$	$+$	$-$	$+$	3
b	$+$	$-$	$-$	$-$	1

Number of roots in [a,b] = $\mathrm{N(a)} - \mathrm{N(b)}$

= 3 – 1 = 2

Bisection Method to Find Root

- Narrowing the range by half